Mathematical Analysis of Fuzzy Reasoning

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Abstract –We have used various kinds of fuzzy reasoning methods for the fuzzy control and the soft science such as "Min-Max Method", "Product- Sum-Gravity Method", "Simplified Fuzzy Reasoning" and so on. In applying the fuzzy reasoning to the fuzzy control and the soft science, it is very important whether the method we choose satisfies monotonicity. However the numerical reasoning functions of these methods do not always hold the monotonicity.

In this paper, by defining some fuzzy mathematical concepts, we obtain some sufficient conditions for the monotonicity of "Product- Sum- Gravity Method" and other similar methods.

1. Introduction

In applying the fuzzy reasoning to the fuzzy control and the soft science, "Min - Max method", "Product - Sum - Gravity Method" and "Simplified Fuzzy Reasoning" are often used. However these methods do not always hold the monotonicity under any conditions. Following is an example that the "Min – Max method" do not hold the monotonicity.
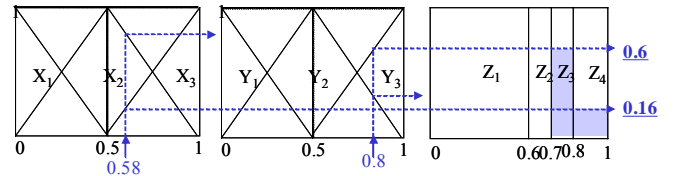
【**Example 1**】

Let the fuzzy sets X_n (n=1,2,3) , Y_n (n=1,2,3) and Z_n (n=1,2,3,4) be given as follows;
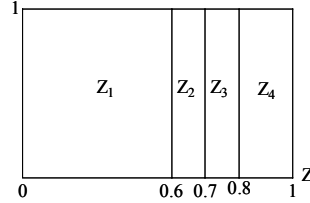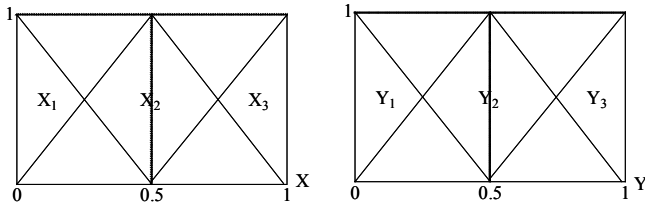
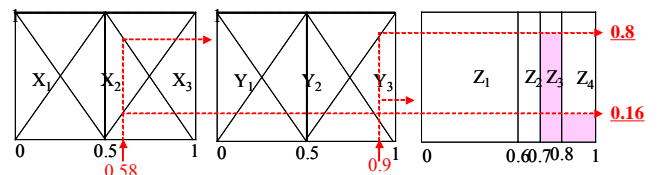And let the fuzzy rule be given as follows;

	Y_1	Y_2	Y_3
X_1	Z_1	Z_2	Z_2
X_2	Z_2	Z_3	Z_3
X_3	Z_2	Z_4	Z_4

Then, if the fact (x, y) = (0.58, 0.80), the result is as follows;

$z_0 = (0.75 \times 0.6 + 0.9 \times 2 \times 0.16) / (0.6 + 2 \times 0.16) = \underline{\textbf{0.8201}}$

However, if y is increased from 0.80 to 0.90, the result is as follows;

$z_0 = (0.75 \times 0.8 + 0.9 \times 2 \times 0.16) / (0.6 + 2 \times 0.16) = \underline{\textbf{0.7928}}$

Therefore, the result z is decreased, in spite that the fact (x, y) is increased. ¶

Above example tells us "Min – Max method" do not always hold the monotonicity under any conditions, in spite that monotnicity is a very important property in using the fuzzy reasoning to various kinds of applications such as educational evaluation, medical diagnosis and so on.

Then, we thought of it very important to know the mathematical sufficient conditions for these fuzzy reasoning methods to hold monotonicity.

2. Fuzzy Mathematical Concepts

In this section, we will define some fuzzy mathematical concepts for the preparation for discussing the monotonicity of the fuzzy reasoning.

【Definition1】 Fuzzy Partitioned Space

Let I be a bounded closed interval on R and let FP(I)={A_i : i∈Λ} be a family of normal fuzzy sets on I. Then, if FP(I) satisfies the following equation, we call FP(I) a fuzzy partition of I, I a fuzzy partitioned space and each A_i, which is element of FP(I), a fuzzy partition set.

$$\Sigma_{i \in \Lambda} A_i(x) = 1, x \in I \qquad \P$$

【Definition2】 α-level Sets

Let A be a fuzzy set on X. Then, the α-level set $[A]^{\alpha}$ is defined with the following equation;

$$[A]^{\alpha} = \begin{cases} \{x \in X\} \mid A(x) \geq \alpha \}, & 0 < \alpha \leq 1 \\ \overline{\{x \in X\} \mid A(x) > 0\}}, & \alpha = 0 \end{cases}$$

where $\overline{s}$ is a closer of s. $\qquad \P$

【Definition3】 Ordered Fuzzy Partition Sets

For two normal convex fuzzy sets A and B defined on closed interval I on R, let α-level sets of A and B be given as follows;

$$\overline{[A]}^{\alpha} = [a_{\alpha}, b_{\alpha}]$$

$$\overline{[B]}^{\alpha} = [c_{\alpha}, d_{\alpha}]$$

Then, if the following equation is satisfied,

$$a_{\alpha} \leq c_{\alpha}, b_{\alpha} \leq d_{\alpha} \; ; \; \alpha \in [0, 1]$$

there is an order relation between A and B, and we express the order relation between A and B as follows;

$$A < B \qquad \P$$

3. Fuzzy Reasoning on the Fuzzy Partitioned Space

In this section, we will discuss the fuzzy reasoning and the definition of the monotonicity of fuzzy rule.

Fuzzy Reasoning on the Fuzzy Partioned Space

Let FP(F_1), FP(F_2), …, FP(F_n), FP(E) be the families of the ordered fuzzy partition sets of fuzzy partitioned space F_1, F_2, …, F_n and E, respectively.

$$FP(F_1) = \{L_{11}, L_{12}, …, L_{1\,m1} \mid L_{11} < L_{12} < …L_{1\,m1}\}$$
$$FP(F_2) = \{L_{21}, L_{22}, …, L_{2\,m2} \mid L_{21} < L_{22} < …L_{2\,m2}\}$$
$$…$$
$$FP(F_n) = \{L_{n1}, L_{n2}, …, L_{n\,mn} \mid L_{n1} < L_{n2} < …L_{n\,mn}\}$$
$$FP(E) = \{A_1, A_2, …, A_r \mid A_1 < A_2 < …<A_r\}$$

And, let the fuzzy rule be given by an onto mapping h;

$$h : FP(F_1) \times FP(F_2) \times …\times FP(F_n) \to FP(E)$$

which we call "Rule Mapping".

Then, we call it "Numerical Fuzzy Reasoning" to solve the conclusion E = $\varphi(x_1, x_2, …, x_n)$ to the fact (x_1, x_2, …, x_n) and also call the function φ "Numerical Reasoning Function". $\qquad \P$

【Definiton4】 Monotonicity of the Fuzzy Rules

Let the rule mapping h be given as follows;

$$h(L_{1\,i1}, L_{2\,i2}, …, L_{n\,in}) = A_p, \qquad \text{--- (1)}$$
$$h(L_{1\,j1}, L_{2\,j2}, …, L_{n\,jn}) = A_q, \qquad \text{--- (2)}$$

where $1 \leq i_1, j_1 \leq m_1, 1 \leq i_2, j_2 \leq m_2, \ldots, 1 \leq i_n, j_n \leq m_n$.

Then, for the equation (1) and (2), if the following equation is satisfied, the rule mapping h is monotone increasing;

$$i_1 \leq j_1, i_2 \leq j_2, \ldots, i_n \leq j_n, \Rightarrow p \leq q$$

Especially, if the following equation is satisfied, the rule mapping h is strictly monotone increasing;

$$\exists k \quad \text{s.t.} \quad i_k < j_k \quad \Rightarrow \quad p < q \qquad ¶$$

3. Monotonicity of the fuzzy reasoning on the fuzzy partitioned space

In this section, we will discuss the sufficient conditions for the fuzzy reasoning defined on the fuzzy partitioned space to satisfy the monotonicity.

3.1 Product - Sum - Gravity Method

Theorem 1

Monotonicity of Product-Sum-Gravity method

Let $\varphi_{PSG} : X_1 \times X_2 \rightarrow Y$ be a numerical fuzzy reasoning function using the Product – Sum – Gravity method and h : $FP(F_1) \times FP(F_2) \rightarrow FP(E)$ be the rule mapping of this fuzzy reasoning.

Then, if the rule mapping h is monotone increasing, the numerical fuzzy reasoning function φ_{PSG} is also monotone increasing.

And, if the rule mapping h is strictly monotone increasing, the numerical fuzzy reasoning function φ_{PSG} is also strictly monotone increasing. ¶

However, for the numerical fuzzy reasoning function using Product–Sum–Gravity method which has more than 3 inputs $\varphi_{PSG} : X_1 \times X_2 \times \ldots \times X_n \rightarrow Y$, above property is not always satisfied.

Example 2

Let $FP(L)=\{L_1, L_2, L_3 \mid L_1 < L_2 < L_3\}$, $FP(M)=\{M_1, M_2, M_3 \mid M_1 < M_2 < M_3\}$, $FP(N)=\{N_1, N_2, N_3 \mid N_1 < N_2 < N_3\}$ and $FP(A)=\{A_1, A_2, A_3, A_4 \mid A_1 < A_2 < A_3 <$ $A_4\}$ be given as follows;
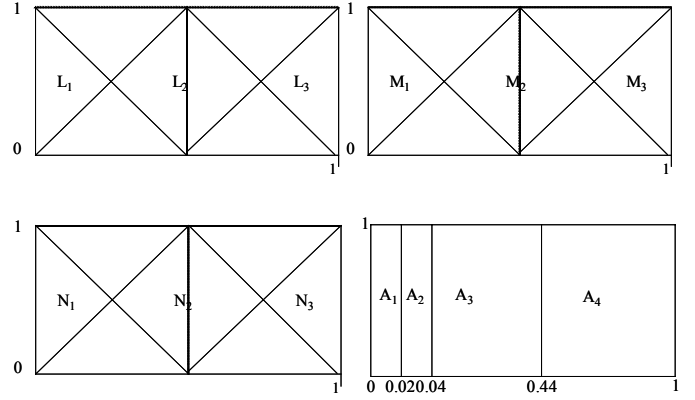

And let the fuzzy rule, which satisfies the monotonicity, be given as follows;

1. L1, M1, N1 ⇒ A1	10. L2, M1, N1 ⇒ A2	19. L3, M1, N1 ⇒ A3
2. L1, M1, N2 ⇒ A1	11. L2, M1, N2 ⇒ A2	20. L3, M1, N2 ⇒ A3
3. L1, M1, N3 ⇒ A1	12. L2, M1, N3 ⇒ A2	21. L3, M1, N3 ⇒ A3
4. L1, M2, N1 ⇒ A1	13. L2, M2, N1 ⇒ A2	22. L3, M2, N1 ⇒ A3
5. L1, M2, N2 ⇒ A1	14. L2, M2, N2 ⇒ A2	23. L3, M2, N2 ⇒ A4
6. L1, M2, N3 ⇒ A2	15. L2, M2, N3 ⇒ A3	24. L3, M2, N3 ⇒ A4
7. L1, M3, N1 ⇒ A1	16. L2, M3, N1 ⇒ A2	25. L3, M3, N1 ⇒ A3
8. L1, M3, N2 ⇒ A3	17. L2, M3, N2 ⇒ A3	26. L3, M3, N2 ⇒ A4
9. L1, M3, N3 ⇒ A3	18. L2, M3, N3 ⇒ A4	27. L3, M3, N3 ⇒ A4

Then, if the fact (l, m, n) = (0.25, 0.75, 0.75), the result is as follows;
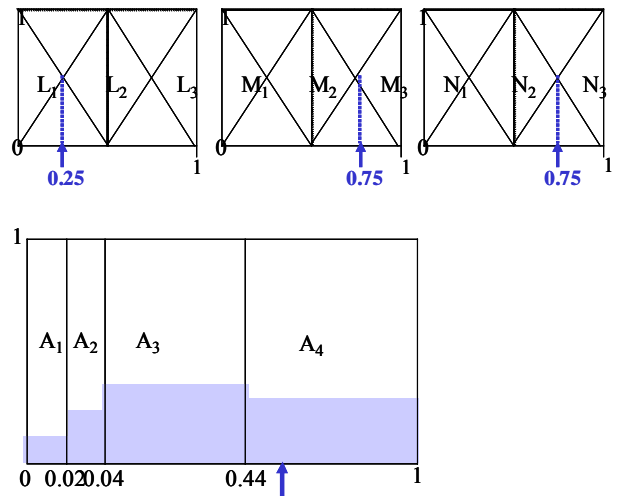
$z_0 = (0.125 \times 0.01 + 0.25 \times 0.03 + 7.5 \times 0.24 + 7 \times 0.72) / (0.125 + 0.25 + 7.5 + 7) = \underline{\mathbf{0.460}}$

However, if l is increased from 0.25 to 0.50, the result is as follows;
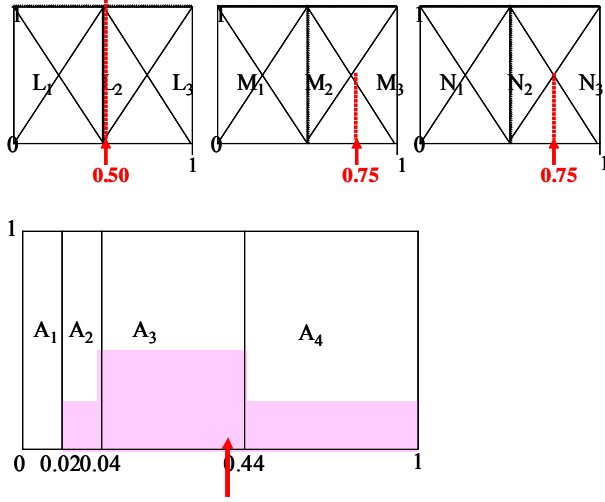
$z_0 = (0.25 \times 0.03 + 10 \times 0.24 + 7 \times 0.72) / (0.25 + 10 + 7)$

= **0.408**

Therefore, the result z is decreased, in spite that the fact (l, m, n) is increased. ¶

For the Product–Sum–Gravity method which has 3 inputs, the additional condition is needed to satisfy monotoniciy.

Theorem 2

Monotonicity of Product-Sum-Gravity method

Let $\varphi_{PSG} : X_1 \times X_2 \times X_3 \to Y$ be a numerical fuzzy reasoning function using the Product–Sum–Gravity method. And let $h : FP(F_1) \times FP(F_2) \times FP(F_3) \to FP(E)$ be the rule mapping of this fuzzy reasoning and be monotone increasing.

Then, if the following condition is satisfied, the numerical fuzzy reasoning function φ_{PSG} is monotone increasing.

$$|P_k(\lambda)| \times |Q_k(\lambda+1)| \geq |P_k(\lambda+1)| \times |Q_k(\lambda)|$$

where

$P_k(\lambda) = \{ \tau_1, \ \tau_2, \ \tau_3, :$
$\qquad h(\ldots, H_\tau, \ldots) = A_\lambda, \ \tau_i = \tau_{i'}, \ \tau_{i'}+1 \}$
$Q_k(\lambda) = \{ \tau_1, \ \tau_2, \ \tau_3, :$
$\qquad h(\ldots, H_\tau+1, \ldots) = A_\lambda, \ \tau_i = \tau_{i'}, \ \tau_{i'}+1 \}$ ¶

3.2 Monotonicity of the fuzzy singleton-type reasoning method

Let $\varphi_{sgl} : X_1 \times X_2 \to Y$ be a numerical fuzzy reasoning function and $h : FP(F_1) \times FP(F_2) \to FP(E)$ be the rule mapping of the fuzzy reasoning. Then, if the FP(E) is given as real number z_i (i=1,2,…,n) which has the weight w_i (i=1,2,…,n), this fuzzy reasoning is called Fuzzy singleton-type reasoning method .

$$FP(E) = \{z_1, z_2, \ldots, z_r \mid z_1 < z_2 < \ldots < z_r \}$$

【**Theorem3**】

Monotoniciy of the Fuzzy Singleton-type Reasoning method

For the Fuzzy Singleton-type reasoning method, if the following conditions are satisfied and the rule mapping h is monotone increasing, the numerical fuzzy reasoning function φ_{sgl} is monotone increasing.

i) $w_1 \geq w_2 \geq \ldots \geq w_r$
ii) $z_i < z_j \Rightarrow w_i / w_j \leq z_j / z_i$ ¶

3.3 Monotonicity of the simplified fuzzy reasoning

For the fuzzy singleton-type reasoning method, if the weight $w_i = 1$ (i=1,2,…,n) then it is called Simplified fuzzy reasoning method.

【**Theorem4**】

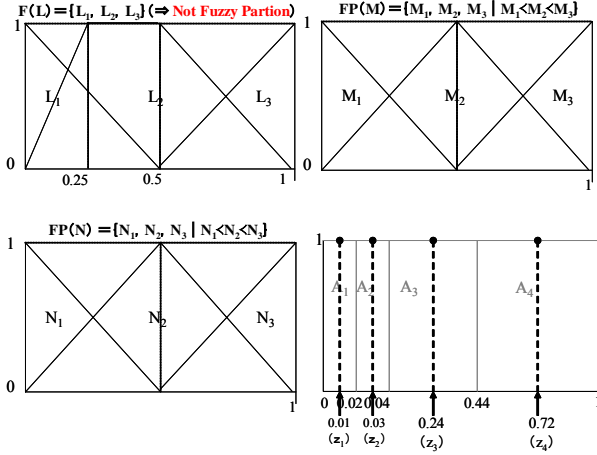**Monotonicity of the Simplified fuzzy Reasoning method**

For the Fuzzy Simplified reasoning method, if the rule mapping h is monotone increasing, the numerical fuzzy reasoning function φ_{smp} is monotone increasing.

Especially, if the rule mapping h is strictly monotone increasing, the numerical fuzzy reasoning φ_{smp} is also strictly monotone increasing. ¶

For the fuzzy singleton-type reasoning method, without the condition of the Fuzzy Partition the numerical fuzzy reasoning function φ_{smp} does not satisfy monotonicity.

【Example 3】

Let F (L)={L$_1$, L$_2$, L$_3$} be a family of the fuzzy sets on X (not the fuzzy partition on X$_1$) and be given as follows. And let FP(M)={M$_1$, M$_2$, M$_3$ | M$_1$ <M$_2$ < M$_3$}, FP(N)={N$_1$, N$_2$, N$_3$ | N$_1$ < N$_2$ < N$_3$} and FP(A)={A$_1$,A$_2$,A$_3$, A$_4$ | A$_1$ < A$_2$ < A$_3$ < A$_4$} be fuzzy partition on X$_2$, X$_3$ and Z respectively, and be given as follows;

And, let the fuzzy rule, which satisfies the monotonicity, be given as follows;

1. L1, M1, N1 ⇒ A1(Z1)
2. L1, M1, N2 ⇒ A1(Z1)
3. L1, M1, N3 ⇒ A1(Z1)
4. L1, M2, N1 ⇒ A1(Z1)
5. L1, M2, N2 ⇒ A1(Z1)
6. L1, M2, N3 ⇒ A2(Z2)
7. L1, M3, N1 ⇒ A1(Z1)
8. L1, M3, N2 ⇒ A3(Z3)
9. L1, M3, N3 ⇒ A3(Z3)

10. L2, M1, N1 ⇒ A2(Z2)
11. L2, M1, N2 ⇒ A2(Z2)
12. L2, M1, N3 ⇒ A2(Z2)
13. L2, M2, N1 ⇒ A2(Z2)
14. L2, M2, N2 ⇒ A2(Z2)
15. L2, M2, N3 ⇒ A3(Z3)
16. L2, M3, N1 ⇒ A2(Z2)
17. L2, M3, N2 ⇒ A3(Z3)
18. L2, M3, N3 ⇒ A4(Z4)

19. L3, M1, N1 ⇒ A3(Z3)
20. L3, M1, N2 ⇒ A3(Z3)
21. L3, M1, N3 ⇒ A3(Z3)
22. L3, M2, N1 ⇒ A3(Z3)
23. L3, M2, N2 ⇒ A4(Z4)
24. L3, M2, N3 ⇒ A4(Z4)
25. L3, M3, N1 ⇒ A3(Z3)
26. L3, M3, N2 ⇒ A4(Z4)
27. L3, M3, N3 ⇒ A4(Z4)
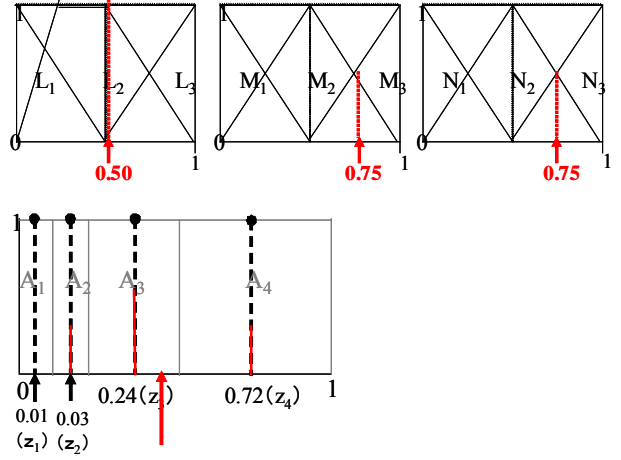
Then, if the fact (l, m, n) = (0.25, 0.75, 0.75), the result is as follows;
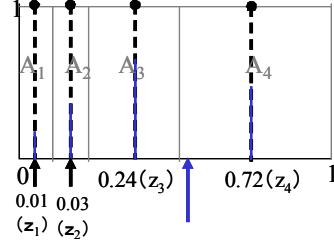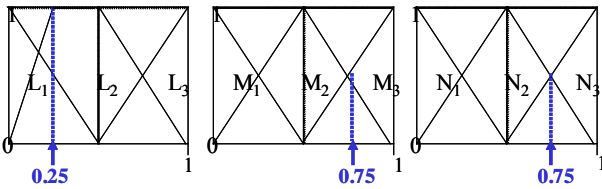
$z_0 = (0.125 \times 1 + 0.375 \times z_2 + 0.625 \times z_3 + 0.375 \times z_4)$
$/ (0.125 + 0.375 + 0.625 + 0.375) = \mathbf{\underline{0.5508}}$

However, if l is increased from 0.25 to 0.50, the result is as follows;

$z_0 = (0.25 \times z_2 + 0.5 \times z_3 + 0.25 \times z_4) /$
$(0.25 + 0.5 + 0.25) = \mathbf{\underline{0.3075}}$

Therefore, the result z is decreased, in spite that the fact (l, m, n) is increased. ¶

4. Conclusion

In this paper, we have discussed the problem of monotonicity which the reasoning function have to have in applying to the various kinds of fields and some sufficient conditions to solve this problem by defining some fuzzy mathematical concepts.

Concretely, for the Product-Sum-Gravity method, Singleton-type method and Simplified method, we have showed the sufficient conditions on which each reasoning functions satisfies the monotonicity by introducing the concept of Fuzzy Partition.

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