Intelligent Nonlinear Control for Underactuated Pendulum Systems

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Abstract—This study aims at establishing an effective intelligent nonlinear control method for underactuated pendulum systems. In the method, a stabilization controller to stabilize a pendulum near unstable equilibrium point and a destabilization controller to input enough energy to transfer it from an arbitrary equilibrium point to a desired unstable one into the system are designed. In addition, an integrator neural network is prepared in parallel with these controllers. The integrator switches several controllers autonomously and adequately based on the system states. Moreover, the integrator integrates them and generates new control force to achieve the control objective. In this study, the adjustment of parameters of the integrator is performed by using the genetic algorithm. The proposed method is applied to an equilibrium point transfer and stabilization control of a double pendulum that possesses four equilibrium points. In order to verify the effectiveness of the proposed method, computational simulations and experiments using a real apparatus were carried out. As a result, it was demonstrated that the integrated intelligent controllers can transfer and stabilize the double pendulum from an arbitrary equilibrium point to a desired unstable one without touching the cart position limit.

I. INTRODUCTION

In the last few years, there has been major interest in developing stabilizing algorithms for practical underactuated systems, such as walking robots, helicopters, satellites, ships, underwater vehicles, aircraft, airships, hovercraft, etc.. Although several control strategies based on passivity, Lyapunov theory, feedback linearization, etc. have been developed for the fully actuated case, such techniques do not apply directly to the case of underactuated nonlinear systems. In general, it is not easy to derive a nonlinear control law systematically for such systems. It is thus clear



The motivation of this study is to establish an effective intelligent nonlinear control technique for underactuated pendulum systems as particular classes of underactuated nonlinear systems. As for a nonlinear control such as a transfer control between equilibrium points, there have been few effective and systematic control design approaches widely applicable to such systems. The configuration of the proposed integrated intelligent control system is shown in Fig.1. In the system, an integrator neural network is prepared in parallel with several controllers for each different local purpose. The integrator switches several controllers autonomously and adequately based on the system states. Additionally, the integrator integrates them and generates new control force to achieve the control objective. It is expected that the proposed method enable us to accomplish several control purposes by using less controllers and switching laws. The integrator is specified by some parameters. In this study, the adjustment of these parameters is performed by a probabilistic optimization method utilizing evolution strategies such as the Genetic Algorithm (GA). The proposed method is applied to an equilibrium point transfer and stabilization control of a double pendulum mounted on a cart. Since the double pendulum possesses a stable equilibrium point (Down-Down) and three unstable equilibrium points (Down-Up, Up-Down, Up-Up), as shown in Fig.2, there are nine paths between equilibrium points in



Fig. 1. Integrated Intelligent Control System.



Stable equilibrium pointUnstable equilibrium pointsFig. 2.Equilibrium Point Transfer and Stabilization Control Problem.

this control problem [1]. To achieve these controls, both a transfer control from one equilibrium point to the other in nonlinear region and a stabilization control near the unstable equilibrium point in linear region are required. In this study, only one swing-up controller, three stabilization controllers and three integrator neural networks are designed to accomplish all nine paths. In order to verify the effectiveness of the proposed method, computational simulations and experiments using an experimental setup are carried out.

II. EQUILIBRIUM POINT TRANSFER AND STABILIZATION CONTROL

This study deals with the equilibrium point transfer and stabilization control of the double pendulum from the arbitrary equilibrium point to the desired unstable one. The problem requires effective switching rules from a swinging controller in order to destabilize the double pendulum from the initial equilibrium point to a stabilizing controller so as to stabilize the double pendulum near the desired unstable one. However, it is not easy to determine systematically an effective switching surface between these two laws and to derive the swinging controller theoretically to solve the nonlinear problem of approaching the desired unstable equilibrium. In the case that the available cart track length is unlimited, this control problem was studied in Yamakita and Furuta et al [1][2][3]. In order to accomplish these controls, a number of controllers and switching rules were prepared for each path so far. Twelve controllers and eleven switching rules were required to realize five paths, that is, 1, 2, 4, 6 and 9 in Fig.2. In this study, since a cart and double pendulum system is used, there is an inherent restriction on the cart track length and the magnitude of control force that can be applied. In addition, there are the various uncertainties such as friction, disturbance and so on. Therefore, the control problem dealt with in this study is how to design a robust controller for a strong nonlinear and complicated system of the double pendulum to cope with the physical limitations and the influence of various uncertainties.

III. DOUBLE PENDULUM

The double pendulum on a cart is an underactuated system with three degrees of freedom and one control input. The model of the system is shown in Fig.3. The cart is driven through the rotary nut which is rotated by the DC servo motor through timing belt. The pendulum 1 and the rotary encoder measuring the angle of the pendulum 1 are installed at the cart. The pendulum 2 is jointed at the pendulum 1 through the rotary encoder measuring the angle of pendulum 3 on the top of pendulum 1. Each pendulum is able to rotate freely in the vertical plane. The equations of motion of cart, pendulum 1 and pendulum 2 can be written as follows:

$$(M_c + M_{p1} + M_{p2}) \ddot{x} + (M_{p1}L_1 + M_{p2}L_{p1}) \dot{\theta}_1 \cos \theta_1 + M_{p2}L_2 \dot{\theta}_2 \cos \theta_2 - (M_{p1}L_1 + M_{p2}L_{p1}) \dot{\theta}_1^2 \sin \theta_1 - M_{p2}L_2 \dot{\theta}_2^2 \sin \theta_2 + C_c \dot{x} = F - f_c$$
(1)



| TABL | ΕI |
|------|----|
|------|----|

| PHYSICAL PARAMETERS | | | | | | | |
|---------------------|--|---------------------------------|------------------------------------|--|--|--|--|
| Symb | ol Quantity | | Value | | | | |
| M_c | mass of cart | | 2.824 kg | | | | |
| M_{p1} | mass of pendulum 1 | | 0.264 kg | | | | |
| M_{p2} | mass of pendulum 2 | | 0.054 kg | | | | |
| L_{p1} | length of pendulum 1 | | 0.321 m | | | | |
| L_{p2} | length of pendulum 2 | | 0.194 m | | | | |
| $\dot{L_1}$ | length from joint to a center of mass of p | endulum 1 | 0.215 m | | | | |
| L_2 | length from joint to a center of mass of p | endulum 2 | 0.095 m | | | | |
| I_1 | moment of inertia of pendulum 1 | 3.22×10^{-1} | 3 kg \cdot m ² | | | | |
| I_2 | moment of inertia of pendulum 2 | 1.86×10^{-1} | 4 kg \cdot m ² | | | | |
| f_c | coulomb friction of cart | $45 \text{ kg} \cdot \text{m}/$ | s^2 | | | | |
| C_c | damping coefficient of cart | 177 kg/s | | | | | |
| C_{p1} | damping coefficient of pendulum 1 | 2.67×10^{-1} | 3 kg \cdot m/s ² | | | | |
| C_{p2} | damping coefficient of pendulum 2 | 1.02×10^{-1} | $4 \text{ kg} \cdot \text{m/s}^2$ | | | | |

$$(M_{p1}L_1 + M_{p2}L_{p1}) \ddot{x}\cos\theta_1 + (M_{p1}L_1^2 + M_{p2}L_{p1}^2 + I_1) \ddot{\theta}_1 + M_{p2}L_{p1}L_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + M_{p2}L_{p1}L_2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + (C_{p1} + C_{p2})\dot{\theta}_1 - C_{p2}\dot{\theta}_2 - (M_{p1}L_1 + M_{p2}L_{p1})g\sin\theta_1 = 0$$
(2)

$$M_{p2}L_{2}\ddot{x}\cos\theta_{2} + M_{p2}L_{p1}L_{2}\theta_{1}\cos(\theta_{1} - \theta_{2}) + (M_{p2}L_{2}^{2} + I_{2})\ddot{\theta}_{2} - M_{p2}L_{p1}L_{2}\dot{\theta}_{1}^{2}\sin(\theta_{1} - \theta_{2}) - C_{p2}\dot{\theta}_{1} + C_{p2}\dot{\theta}_{2} - M_{p2}L_{2}g\sin\theta_{2} = 0$$
(3)

All symbols used in the above equations are defined in Fig.3. x(t) is the displacement of the cart. $\theta_1(t)$ and $\theta_2(t)$ indicate the angle of the pendulum 1 and the angle of the pendulum 2, measured positive in a clockwise direction respectively. The control force F is expressed as follow [4]:

$$F = \frac{K_t}{R} d_2 r_g e - \frac{K_t K_e}{R} d_2^2 r_g^2 \dot{x} - \left[(I_m + I_{gm}) r_g^2 + (I_n + I_{gn}) \right] d_2^2 \ddot{x}$$
(4)

where e(t) shows the input voltage of the motor. These dynamical models are nonlinear for the pendulum angle. There is friction between the cart and the fixed ball screw. The physical parameters of experimental equipment are shown in Table I.

IV. DESIGN OF CONTROLLER

The design technique of the proposed method is described as follows. The structure of the proposed integrated intel-



Fig. 4. Structure of Integrated Intelligent Control System.

ligent controller is shown in Fig.4. Firstly, three stabilization controllers are designed individually based on linear models around each unstable equilibrium point. Secondly, one swing-up controller is designed based on the energy of the double pendulum. Finally, the integrator neural network which is prepared in parallel with these controllers is designed.

A. Stabilization Controller

A stabilization controller should stabilize the double pendulum near the unstable equilibrium point. The stabilization controller is realized as a state-space controller with the feedback gain vector \mathbf{k}_i , which is calculated according to the linear-quadratic regulator (LQR) design method. The following control gain vector \mathbf{k}_i and state variables \mathbf{x}_i at each equilibrium point are used.

Down-Up

$$Q_{du} = diag \begin{pmatrix} 0 & 300 & 0 & 1 & 0 & 1 \end{pmatrix}, R_{du} = 1.0$$

$$\mathbf{k}_{du} = \begin{bmatrix} -85.9198 & -17.3205 & 53.2593 & 493.0170 & -65.6931 & -616.7089 \end{bmatrix}$$

$$\mathbf{x}_{du} = \begin{bmatrix} \dot{x} & x & \dot{\theta}_{1d} & \theta_{1d} & \dot{\theta}_{2} & \theta_{2} \end{bmatrix}^{T}$$

$$u_{du} = -\mathbf{k}_{du} \cdot \mathbf{x}_{du} \tag{5}$$

Up-Down

 $Q_{ud} = diag \left(\begin{array}{cccc} 0 & 300 & 0 & 50000 & 0 & 500 \end{array} \right), R_{ud} = 1.0$ $\mathbf{k}_{ud} = \begin{bmatrix} -92.3226 & -17.3205 & -32.4339 & -313.7022 & -0.2756 & 14.3866 \end{bmatrix}$ $\mathbf{x}_{ud} = \begin{bmatrix} \dot{x} & x & \dot{\theta}_1 & \theta_1 & \dot{\theta}_{2d} & \theta_{2d} \end{bmatrix}^T$

$$u_{ud} = -\mathbf{k}_{ud} \cdot \mathbf{x}_{ud} \tag{6}$$

Up-Up

 $Q_{uu} = diag \begin{pmatrix} 0 & 300 & 0 & 1 & 0 & 1 \end{pmatrix}, R_{uu} = 1.0$ $\mathbf{k}_{uu} = \begin{bmatrix} 10.2913 & 17.3205 & 14.0202 & -353.1186 & -64.2297 & 607.4820 \end{bmatrix}$ $\mathbf{x}_{uu} = \begin{bmatrix} \dot{x} & x & \dot{\theta}_1 & \theta_1 & \dot{\theta}_2 & \theta_2 \end{bmatrix}^T$ $u_{uu} = -\mathbf{k}_{uu} \cdot \mathbf{x}_{uu}$ (7)

B. Swing-up Controller

In this study, the swing-up controller which takes care of the physical limitations is designed based on the energy of the double pendulum [5]-[8]. The energy of the uncontrolled double pendulum (u = 0) is written as follows:

$$E = \frac{1}{2} \left(I_1 + M_{p1} L_1^2 + M_{p2} L_{p1}^2 \right) \theta_1^2 + M_{p2} L_{p1} L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + \frac{1}{2} \left(I_2 + M_{p2} L_2^2 \right) \dot{\theta}_2^2 + M_{p1} L_1 g \left(\cos \theta_1 - 1 \right) + M_{p2} g \left[L_{p1} \left(\cos \theta_1 - 1 \right) + L_2 \left(\cos \theta_2 - 1 \right) \right]$$
(8)

The nominal energy is defined to be zero when the double pendulum is in Up-Up equilibrium point. Therefore, the swing-up controller can input enough energy into the system so that it is able to reach the desired equilibrium point from an arbitrary angle of the pendulum.

$$u_{swg} = \alpha \cdot \left\{ \frac{2}{1 + \exp\left[-\gamma \cdot (E - \beta) \cdot \operatorname{sgn}_{x}(\theta_{1})\right]} - 1 \right\}$$
(9)

$$\operatorname{sgn}_{x}(\theta_{1}) = \begin{cases} 1 : \sin \theta_{1} > 0 \operatorname{or} \sin \theta_{1} = 0 \cap x \ge 0\\ -1 : \sin \theta_{1} < 0 \operatorname{or} \sin \theta_{1} = 0 \cap x < 0 \end{cases}$$
(10)

The swing-up controller is specified by three parameters $\{\alpha, \beta, \gamma\}$. α and β indicate the parameters for the limitation of the track length and for compensation of the friction loss of the system respectively. γ represents the parameter of a sigmoid function in order to reduce the chattering. Determination of the values of α , β and γ requires some trial and error and is difficult to do manually. In this study, $\alpha = 16$, $\beta = 0.15$ and $\gamma = 0.15$ are used. The computational simulation for the case that the double pendulum was initialized at stable equilibrium point was carried out. Figure 5 shows the time histories, in order, the displacement of the cart, the angular displacements of the pendulum 1 and the pendulum 2, the control input and the energy of the double pendulum. It was demonstrated as shown in Fig.5 that the designed swing-up controller provides the double pendulum with the energy when the pendulum 1 is in lowest position ($\theta_1 = \pi$), and the energy of the double pendulum is reached to the desired energy. As a result, it was confirmed that the swing-up controller can swing up the double pendulum at approximately 4 s.



Fig. 5. Simulation Result in the case that started from Down-Down.

C. Integrator Neural Network

To transfer and stabilize the double pendulum from the arbitrary equilibrium point to the desired unstable one without touching the physical limitations, these designed controllers should be switched and integrated adequately according to the situation. However, it is not easy to derive theoretically suitable switching and integration rules of several controllers. In the method, as shown in Fig.4, the integrator neural network is prepared in parallel with several local controllers. The integrator switches and integrates four controllers automatically and appropriately based on the system states. Since the double pendulum has three unstable equilibrium points, we design three integrators, *integrator*_{du}, *integrator*_{ud} and *integrator*_{uu}, by using the similar architecture of radial basis function neural network and learning method.

1) Architecture of Integrator: The architecture of the integrator neural network is shown in Fig.6. The integrator adopts the hidden layers which consist of the following radial basis neurons.

$$\varphi_j \left(\mathbf{x}, \mathbf{a}_j, \mathbf{b}_j \right) = \prod_{i=1}^l \exp\left\{ -\frac{\left(\mathbf{x} - \mathbf{a}_j \right)^2}{\mathbf{b}_j^2} \right\}$$
 (11)

where, **x** is the input vector with elements x_i , and \mathbf{a}_j is the vector determining the center of basis function φ_j and has elements a_{ji} . \mathbf{b}_j represents the vector determining the width of the Gaussian and has elements b_{ji} . l is the number



of nodes in the input layer. The following sharply sigmoid function r_k is used as an activation function of the output node k.

$$r_k(\mathbf{x}) = \frac{1}{1 + \exp\left(-200\left(\sum_{j=1}^m w_{kj} \cdot \varphi_j(\mathbf{x}) + w_{k\theta}\right)\right)}$$
(12)

where, w_{kj} is a weight connecting the ontput node k to the hidden node j and $w_{k\theta}$ is a bias weight. m is the number of nodes in the hidden layer. r_k varies continuously between 0 and 1 and means the degree of importance for each controller. In other words, it can be expected that adjustment of each control gain is performed according to the situation. The control input u_{total} is defined as the following equation.

$$u_{total} = r_{uu}u_{uu} + r_{ud}u_{ud} + r_{du}u_{du} + r_{swg}u_{swg}$$
(13)

2) Learning Method: The integrator is specified by four adjustable parameters $\{a_{ji}, b_{ji}, w_{kj}, w_{k\theta}\}$. In this study, tuning of them is carried out based on the GA. These parameters are regarded as GA parameters and GAs begin a set of thirty randomly generated states, called the population. During training GA process, (a) initial population, (b) fitness function, (c) selection, (d) crossover and (e) mutation, the individual chromosome to perform the control objective is searched. To calculate the fitness value, the following fitness function is prepared.

$$f_{fit} = \frac{h(x)}{n} \cdot \sum_{t=0}^{n} \left(\frac{1}{10 \cdot p_1^2(t) + 10 \cdot p_2^2(t) + 1} \right)$$
(14)

$$h(x) = \begin{cases} 1 & |x| \le 0.4 \text{ m} \\ 0.1 & |x| > 0.4 \text{ m} \end{cases}$$
(15)

where, $n = t_f/\Delta t$. t_f and Δt represent the simulation period and the sampling period respectively. In this study, $t_f = 10$ s and $\Delta t = 5$ ms are used. p_1 and p_2 are set out as shown in Table II. The fitness value is the highest when the double pendulum is in the desired unstable equilibrium point. In our practical facility, the movable track length of the cart is limited. h(x) indicates the penalty function on the cart position limit. When the cart reaches the end of the track, the fitness value is low. In the particular case from Up-Down to Up-Up, the following fitness function in consideration of energy variation is used to swing-up and stabilize the double pendulum without falling down.

TABLE II

| GA PARAMETERS. | | | | | | | | |
|-------------------|---------------|---------------|--|------------|-----------|--|--|--|
| desired | p_1 | p_2 | j_1 :initial states (θ_1, θ_2) | | | | | |
| equilibrium point | | | $j_1 = 0$ | $j_1 = 1$ | $j_1 = 2$ | | | |
| Down-Up | θ_{1d} | θ_2 | (π,π) | $(0,\pi)$ | (0,0) | | | |
| Up-Down | θ_1 | θ_{2d} | (π,π) | $(\pi, 0)$ | (0,0) | | | |
| Up-Up | θ_1 | θ_2 | (π,π) | $(\pi, 0)$ | $(0,\pi)$ | | | |

$$f_{fit} = \frac{h(x) \cdot k(\theta_1)}{n} \cdot \sum_{t=0}^{n} \left(\frac{1}{10 \cdot \theta_1^2(t) + 10 \cdot \theta_2^2(t) + 1} \right)$$
(16)

$$k(\theta_1) = \begin{cases} 1 & |\theta_1| \le 0.2 \text{ rad} \\ 0.01 & |\theta_1| > 0.2 \text{ rad} \end{cases}$$
(17)

where $k(\theta_1)$ indicates the penalty function on the pendulum 1 angle. If once the double pendulum is falling down, the fitness value is low. In the cases from three initial conditions shown in Table II, the above-mentioned fitness function is calculated and then the sum of them is regarded as the fitness value of the individual chromosome.

$$f_{fit_total} = \sum_{j_1=0}^{2} f_{fit}(j_1)$$
(18)

During training the GA process, one individual chromosome whose fitness value is maximum is chosen as the integrator neural network.

V. SIMULATION RESULT

In order to verify the performance of the proposed method, simulations for the cases from the arbitrary equilibrium points to each unstable equilibrium point were carried out. From the results, it was confirmed that the proposed controller can achieve all nine paths among equilibrium points by switching and integrating four controllers autonomously and adequately according to situations.

VI. EXPERIMENTAL RESULT

In addition, experiments using a real apparatus were carried out. The initial states of \dot{x} , x, $\dot{\theta_1}$ and $\dot{\theta_2}$ are zero. In this study, the length of the track is 0.8m. The servo motor range is ± 25 V. The displacement of the cart and the angular displacements of the pendulum 1 and the pendulum 2 are observed at the intervals of 5ms, that is, the sampling period. Their velocities are calculated from the difference between sequential displacements. Figures 7 to 11 show the time histories, in order, the displacement of the cart, the angular displacements of pendulum 1 and pendulum 2, the control input and the outputs of the integrator.

A. Stable Equilibrium Point→ Unstable Equilibrium Point

Figures 7 to 9 show the result of the experiments when it was initialized at the stable equilibrium point, that is, Down-Down respectively. It was demonstrated as shown in Fig.7 that the controller can transfer the double pendulum from Down-Down by integrating Down-Up with Up-Up stabilization controllers and then stabilize it near Down-Up by switching only the Down-Up stabilization controller. As shown in Figs. 8 and 9, the proposed controller can transfer and stabilize the double pendulum from Down-Down to near the desired unstable equilibrium point respectively. Fig.12(a) shows the view of the double pendulum on a cart during swinging up and stabilizing control for the case that started from Down-Down to Up-Up.

B. Unstable Equilibrium Point \rightarrow Unstable Equilibrium Point

Figures 10 and 11 show the result of the experiments for the transfer and stabilization control between two unstable equilibrium points. Figure 10 shows the result of the experiment when it was initialized at Up-Up. It can be seen from Fig.10 that the controller can transfer the double pendulum from the Up-Up to Up-Down without falling down by integrating Up-Down stabilization controller with the swing-up controller and then stabilize it near the Up-



Fig. 7. Experimental Result from Down-Down to Down-Up.



Fig. 8. Experimental Result from Down-Down to Up-Down.



Fig. 9. Experimental Result from Down-Down to Up-Up.



Fig. 10. Experimental Result from Up-Up to Up-Down.

Down equilibrium point by switching only Up-Down stabilization controller. It was demonstrated as shown in Figs. 11 and 12(b) that the proposed controller can transfer and stabilize the double pendulum from Up-Down to Up-Up without falling down by switching and integrating four controllers adequately.

VII. CONCLUSION

In this paper, an effective intelligent nonlinear control method for underactuated nonlinear systems was presented. To cope with the increases of number of controllers and switching rules of several controllers for such systems, the integrator neural network which switches and integrates several controllers based on the system states was proposed. For the application to the equilibrium point transfer and stabilization control of the double pendulum which has four equilibrium points, the integrator which takes consideration into the physical limitations was designed. In order to verify the effectiveness of the proposed method, computational simulations and experiments on a real facility were carried out. As a result, it was confirmed that the proposed controller can transfer and stabilize the double pendulum from the arbitrary equilibrium point to the desired unstable one. From the simulations and the experiments, it was demonstrated that the integrated intelligent control method is useful for underactuated pendulum systems.

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Fig. 11. Experimental Result from Up-Down to Up-Up.



(a) From Down-Down to Up-Up. (b) From Up-Down to Up-Up.
 Fig. 12. Experimental Result.

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