An Additive Utility for Choices of Income Streams

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***Abstract*— This paper provides an additive utility model, which is expressed as a weighted sum of several von Neumann-Morgenstern utilities, for decision problems of income streams. First, it is shown that the values of weights for periods play a role differently from the shapes of utilities, in comparison with another additive model expressed as a weighted sum of one von Neumann-Morgenstern utility. Secondly, by introducing the concept of joint receipt into decision problems of this sort, we make it possible to consider choices between income streams with different numbers of periods.**

I. Introduction

Several revisions [2], [4] of an additive utility model (e.g., in the context of conjoint measurement) had been provided for decision problems of temporal sequences of outcomes, such as income streams. The reason is not only that these problems are important but also that the additive model can easily accept the concept of weights for different periods and so enables us to deal with the effect of temporal discounting. Let $(g_1, \ldots, g_n)$ denote an n-year salary, where each g_i $(i = 1, \ldots, n)$ is a money lottery (i.e., an incentive wage) or a monetary outcome for the i-th year. This paper presupposes a weighted additive utility model written in the form

$$U(g_1, \ldots, g_n) = \sum_{i=1}^{n} \pi_i u_i(g_i),$$

where the u_i are utility functions of von Neumann and Morgenstern [3] and π_i are positive real numbers, interpreted as weights of periods. But the most typical weighted utility model [2], [4] is of the form

$$U(g_1, \ldots, g_n) = \sum_{i=1}^{n} \pi_i u(g_i).$$

That is, one utility function u is used in this model. The first aim of this paper is to verify the explanation ability of our additive model (displayed earlier), in comparison with the typical additive one (displayed later). In other words, we will show that the role of the π_i is different from that of the u_i.

In general, these additive models consider only choices between income streams with the same length of periods, say with n periods. So the second aim of this paper is to propose a framework so that income streams with the distinct length of periods, say with $n-1$ periods and with n periods ($n \geq 2$), can be compared. The concept of joint receipt, which was proposed by Luce [6], will be useful for this purpose. By using the operation $\oplus$ of joint receipt, we can describe the situation of receiving two or more uncertain alternatives simultaneously. For example, the n-year salary $(g_1, \ldots, g_n)$ is written as $g_1 \oplus \cdots \oplus g_n$. Also, many utility representations [6], [9] have already been proposed for $\oplus$ having different algebraic properties. But these representations, without a substantial revision, cannot reflect the temporal effect on the decision of income streams explicitly. Therefore we adhere to the weighted additive model, but are willing to utilize an algebraic property of $\oplus$ so as to attain the second aim. By assuming the existence of an element e acting like a right weak identity, i.e., $g_1 \oplus \cdots \oplus g_{n-1} \sim g_1 \oplus \cdots \oplus g_{n-1} \oplus e$ (where $\sim$ is an indifference relation), it is possible to identify $g_1 \oplus \cdots \oplus g_{n-1}$ with $g_1 \oplus \cdots \oplus g_{n-1} \oplus e$, and hence two elements $g_1 \oplus \cdots \oplus g_{n-1}$ and $h_1 \oplus \cdots \oplus h_n$ become comparable, where $h_1 \oplus \cdots \oplus h_n$ is another n-year salary.

This paper is structured as follows. Section II gives basic concepts and notations, in which the von Neumann-Morgenstern theory and axioms of the joint receipt operation are included. Section III shows the uniqueness property of our additive model (which is important for measurement of utilities) and the axiom that enables us to derive the typical additive model from our additive model. Section IV compares the explanation ability of our additive model with that of the typical additive model through a decision problem of two-year salaries, and also considers choices between one- and two-year salaries. Section V contains conclusions. The proofs of the propositions in Section III can be found in the Appendix.

II. Concepts and Notations

A. The von Neumann-Morgenstern theory

Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers. Let $X = \{x_1, \ldots, x_m\}$, with m a fixed positive integer, be a finite set of monetary outcomes. A (*simple*) *lottery* is a probability distribution on X, i.e., $\sum_{i=1}^{m} p(x_i) = 1$, $p(x_i) \geq 0$ for all $i = 1, \ldots, m$. Let $1_{\{x_i\}}$ denote a lottery such that $p(x_i) = 1$ for $x_i \in X$. Every lottery g is expressed as a convex combination $\sum_{i=1}^{m} p_i 1_{\{x_i\}}$. The following expression is also used:

$$g = (x_1, p_1; \ldots; x_m, p_m).$$

Let $\mathcal{G}$ be the set of all lotteries on X. Clearly $\mathcal{G}$ is a nonempty convex set.

Let $\succcurlyeq$ be a binary relation (preference or indifference) on $\mathcal{G}$. The strict preference $\succ$ on $\mathcal{G}$ and the indifference $\sim$ on $\mathcal{G}$ are defined as follows: for all $g,\ h \in \mathcal{G}$,

$$\begin{aligned} g \succ h \quad &\text{if and only if} \quad g \succcurlyeq h \text{ and not } h \succcurlyeq g, \\ g \sim h \quad &\text{if and only if} \quad g \succcurlyeq h \text{ and } h \succcurlyeq g. \end{aligned}$$

$\preccurlyeq$ and $\prec$ denote the reversed relations of $\succcurlyeq$ and $\succ$. The binary relation $\succcurlyeq$ on $\mathcal{G}$ is a *weak order* if and only if it is connected ($g \succcurlyeq h$ or $h \succcurlyeq g$ for all $g,\, h \in \mathcal{G}$) and transitive ($f \succcurlyeq g$ and $g \succcurlyeq h \Rightarrow f \succcurlyeq h$ for all $f,\, g,\, h \in \mathcal{G}$). Let $\succcurlyeq$ be a weak order on $\mathcal{G}$. The set $\mathcal{G}$ is *bounded above* (resp., *below*) in $\succcurlyeq$ if there exists a largest element (resp., a smallest element) in $\mathcal{G}$, i.e., an element $h \in G$ such that $h \succcurlyeq g$ (resp., $g \succcurlyeq h$) for all $g \in G$. The set $\mathcal{G}$ is *unbounded* in $\succcurlyeq$ when it is neither bounded above nor bounded below.

A binary relation $\succcurlyeq$ on $\mathcal{G}$ satisfies *independence* if $f \succ g \Rightarrow \lambda f + (1-\lambda)h \succ \lambda g + (1-\lambda)h$ for all $f,\ g,\ h \in \mathcal{G}$ and all $0 < \lambda < 1$ in $\mathbb{R}$. $\succcurlyeq$ is *continuous* if, for all $f \succ g$ and all $g \succ h$, there exist $0 < \lambda,\, \mu < 1$ such that $\lambda f + (1-\lambda)h \succ g$ and $g \succ \mu f + (1-\mu)h$. If $\succcurlyeq$ is a weak order and satisfies independence and continuity, then it is said that the *von Neumann-Morgenstern axioms hold*. The von Neumann-Morgenstern axioms are necessary and sufficient for the existence of a real-valued function u on $\mathcal{G}$ having the properties: for all $g,\ h \in \mathcal{G}$ and all $0 \le \lambda \le 1$,

$$\begin{aligned} g \succcurlyeq h \quad &\Leftrightarrow \quad u(g) \ge u(h), \\ u(\lambda g + (1-\lambda)h) \quad &= \quad \lambda u(g) + (1-\lambda)u(h). \end{aligned}$$

The former is the *order-preserving property* and the latter is the *linearity property* (in the convexity operation). Such a u is unique up to a positive affine transformation i.e., $u \to \alpha u + \beta,\ \alpha > 0,\ \beta \in \mathbb{R}$. Let $\succcurlyeq'$ be also a binary relation on $\mathcal{G}$ satisfying the von Neumann-Morgenstern axioms, and let u' be a function having the two properties. If $g \succcurlyeq h \Leftrightarrow g \succcurlyeq' h$ for all $g,\, h \in \mathcal{G}$, then it follows that $u' = \alpha u + \beta$. Two utilities connected in this way are called *equivalent*. See, for example, [5] for a comprehensive survey of the von Neumann-Morgenstern theory.

Henceforth a function satisfying the order-preserving and linearity properties is referred to as a *von Neumann-Morgenstern utility*. This function is extended from $\mathcal{G}$ to X by defining the utility of outcome x to be the utility of the degenerate lottery $1_{\{x\}}$: $u(x) = u(1_{\{x\}})$ for all $x \in X$. Then the linearity property leads a von Neumann-Morgenstern utility to the expected-utility form for all simple lotteries $g = \sum_{i=1}^{m} p_i 1_{\{x_i\}} \in \mathcal{G}$:

$$u(g) = \sum_{i=1}^{m} p_i u(x_i). \tag{1}$$

When a decision maker's utility over monetary outcomes, i.e., $u(x)$ defined above, is (strictly) concave, his or her preference is said to be (*strictly*) *risk averse*. When $u(x)$ is (strictly) convex, the decision maker's preference is said to be (*strictly*) *risk seeking*. Here the adverb "strictly" is used to exclude the preference based on a mere expectation, i.e., $u(g) = E(g)(= \sum_{i=1}^{m} p_i x_i)$. Fig. 1 shows a strictly concave utility by the black line and a strictly convex utility by the gray line. Henceforth in this paper, we use the terms "risk averse" or "risk seeking" to mean strictly risk averse or strictly risk seeking. It is seen that either

$$u(g) < u(E(g)) \quad \text{or} \quad u(g) > u(E(g))$$

holds depending on whether the preference is risk averse or risk seeking.

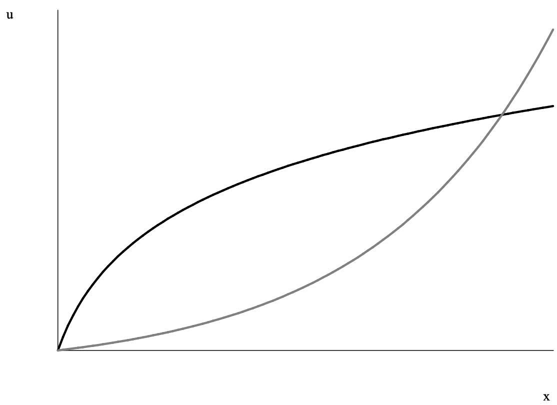

Fig. 1. A concave utility and a convex utility.

B. Joint-receipt semigroup

A *joint receipt operation* $\oplus$ is an operation joining two or more lotteries. For example, $g \oplus h$ indicates the simultaneous receipt of two lotteries g and h, called their *joint receipt*. Note that outcomes of lotteries in each joint receipt do not always appear at the same time. Let G be a set such that the joint receipt operation $\oplus$ is a binary operation on it, i.e., $\oplus$ is a map of $G \times G$ into G. Assume that G is endowed with a weak order $\succsim$. In this paper, joint receipts of degenerate lotteries substitute for those of pure outcomes, i.e., $1_{\{x\}} \oplus 1_{\{y\}}$ $(x,\ y \in X)$ substitutes for $x \oplus y$. This substitution for joint receipts of pure outcomes is not used in Luce's [6] approach. But his axioms guarantees that $1_{\{x\}} \oplus 1_{\{y\}}$ is indifferent to $x \oplus y$.

Definition 1: $\langle G,\ \succsim,\ \oplus \rangle$ is said to be a joint-receipt semigroup if and only if for all $g,\ h,\ f,\ f' \in G$,

A1. Weak order: $\succsim$ is a weak order.
A2. Weak associativity: $(f \oplus g) \oplus h \sim f \oplus (g \oplus h)$.
A3. Weak monotonicity:

$$g \succsim h \ \Leftrightarrow\ f \oplus g \oplus f' \succsim f \oplus h \oplus f'.$$

In particular, if weak commutativity holds, i.e., if $g \oplus h \sim h \oplus g$, then a joint-receipt semigroup $\langle G,\ \succsim,\ \oplus \rangle$ is said to be commutative; otherwise it is said to be non-commutative.

Cho and Luce [10] gave empirical evidence supporting the commutativity of $\oplus$ for money gains or losses by their experiments. But Luce [6] suggested that it might be wise to consider addition rules different from $x \oplus y = x + y$;

Marchant and Luce [9] explored the axioms so as to define $\oplus$ as generalized additions, which includes the non-commutative case, and specified the forms of utility functions. Moreover, Luce [6] examined the associativity of $\oplus$ for pure outcomes, by comparing behavior of utility models related to empirical data [7]. This paper assumes that $\oplus$ is weakly associative but not commutative. The reason is as follows. We shall now interpret $\oplus$ in the context of "making an agreement." Let $x > y$ and z be positive real numbers. Let $x \oplus y$ denote to make the agreement of $\$x$ this year and $\$y$ next year as a two-year salary. Usually, $x \oplus y$ will not be indifferent to $y \oplus x$. For the associativity, we give the parenthesis the meaning of a prior agreement. Let $(x \oplus y) \oplus z$ mean that a decision maker first reaches the agreement of $\$x$ and $\$y$ for the first two years and then reaches the agreement of $\$z$ for the last year. Similarly, $x \oplus (y \oplus z)$ means that the decision maker reaches the agreement of salaries for the latter two years prior to the agreement for the first year. If the decision maker is informed of the specification of every salary in a joint receipt previously, then he or she will consider $(x \oplus y) \oplus z$ to be indifferent to $x \oplus (y \oplus z)$, because both the joint receipts are actually equivalent contracts.

III. Additive Utility Models

Throughout the rest of the paper, for each $i = 1, \ldots, n,\ n \geq 1$, let $\succcurlyeq_i$ be an unbounded weak order on $\mathcal{G}$ that satisfies the von Neumann-Morgenstern axioms, and let u_i be a von Neumann-Morgenstern utility on $\langle \mathcal{G},\ \succcurlyeq_i \rangle$ such that $u_i(f_i) = 1,\ u(e_i) = 0$ for fixed $f_i, e_i \in \mathcal{G}$. Let $\mathcal{G}^n = \mathcal{G} \times \cdots \times \mathcal{G}$ (n copies), the i-th $\mathcal{G}$ of which is endowed with $\succcurlyeq_i$. Let $\mathcal{G}^\infty = \bigcup_{n=1}^{\infty} \mathcal{G}^n$ and let $\succsim$ be a weak order on $\mathcal{G}^\infty$. The restriction of $\succsim$ to $\mathcal{G}^n$ is written as $\succsim^n$. Here $\succsim^1$ is equal to $\succcurlyeq_1$. By identifying $(g_1, \ldots, g_n)$ with $g_1 \oplus \cdots \oplus g_n$, one can see that $\oplus$ is a binary operation on $\mathcal{G}^\infty$.

This paper assumes the following additive model for $\langle \mathcal{G}^\infty,\ \succsim,\ \oplus \rangle$: let n be fixed, then there exist positive real numbers $\pi_1, \ldots, \pi_n$ with $\sum_{i=1}^n \pi_i = 1$ such that, for all $g = g_1 \oplus \cdots \oplus g_n,\ h = h_1 \oplus \cdots \oplus h_n \in \mathcal{G}^n$,

$$g \succsim^n h \ \Leftrightarrow\ \sum_{i=1}^{n} \pi_i u_i(g_i) \geq \sum_{i=1}^{n} \pi_i u_i(h_i). \tag{2}$$

The uniqueness property of the weights π_i, along with that of the utilities u_i, is given as follows.

Proposition 2: Assume that the additive representation of Eq. (2) holds. Then the π_i are unique for such u_i as scaled above. Moreover, von Neumann-Morgenstern utilities v_i scaled differently satisfy the representation of Eq. (2) in place of the u_i if and only if there exist real numbers $\alpha > 0$ and β_i such that, for each $i = 1, \ldots, k,\ v_i = \alpha u_i + \beta_i$.

Strictly speaking, the additive form (Eq. (2)) is not a representation of $\langle \mathcal{G}^\infty, \succsim, \oplus \rangle$ but a representation of $\langle \mathcal{G}^n, \succsim^n \rangle$, because the value of each π_i depends on the number n. Recall that our aim is to verify the explanation ability of the additive model of the form of Eq. (2) in decision problems. This paper deals with decision problems of two-year salaries. It is, therefore, sufficient to provide the additive model of Eq. (2) with $n \geq 2$. Indeed, by embedding $\mathcal{G}^k$ in $\mathcal{G}^n$, with $k < n$, one will be able to compare k- and n-year salaries.

In what follows we view the necessary condition for the additive representation of Eq. (2). For this the following notation is used: for $x,\ g_i \in \mathcal{G}$ $(i = 1, \ldots, i-1, i+1, \ldots, n)$,

$$\begin{aligned} g_{\bar{i}} &= g_1 \oplus \cdots \oplus g_{i-1} \oplus g_{i+1} \oplus \cdots \oplus g_n, \\ x \oplus g_{\bar{i}} &= g_1, \oplus \cdots \oplus g_{i-1} \oplus x \oplus\ g_{i+1} \oplus \cdots \oplus g_n. \end{aligned}$$

Clearly A1, A2, A3, and weak non-commutativity are necessary for the representation of the form of Eq. (2). Therefore at least $\langle \mathcal{G}^\infty,\ \succsim,\ \oplus \rangle$ must be a joint-receipt non-commutative semigroup. But A3 is rewritten in such a way that the relationship of each $\succcurlyeq_i$ $(i = 1, \ldots, n)$ to $\succsim$ is made clear:

A3. Weak monotonicity: for every $i = 1, \ldots, n$ and for all $x,\ y \in \mathcal{G}$ and all $g_{\bar{i}} \in \mathcal{G}^{n-1}$,

$$x \succcurlyeq_i y \ \Leftrightarrow\ x \oplus g_{\bar{i}} \succsim^n y \oplus g_{\bar{i}}$$

It is easily seen that there exists an element acting like a right weak identity: for $m > n$, $g \sim g \oplus e_{n+1} \oplus \cdots \oplus e_m$ holds for all $g \in \mathcal{G}^n$. In other words, for all $g,\ h \in \mathcal{G}^n$,

$$\begin{aligned} & g \succsim^n h \\ \Leftrightarrow\ & g \oplus e_{n+1} \oplus \cdots \oplus e_m \succsim^m h \oplus e_{n+1} \oplus \cdots \oplus e_m. \end{aligned}$$

This enables us to identify every $g \in \mathcal{G}^n$ with an element of $\mathcal{G}^m$. Note here that $e_{n+1} \oplus \cdots \oplus e_m$ is not really a weak identity. Indeed, let $h \in \mathcal{G}^k$ with $k \neq n,\ k \geq 1$. Then for $h \oplus e_{n+1} \oplus \cdots \oplus e_m$, each e_{n+i} $(i = 1, \ldots, m-n)$ is evaluated with $\succcurlyeq_{k+i}$ (in general, $u_{k+i}(e_{n+i}) \neq 0$). Hence h is not always $\sim h \oplus e_{n+1} \oplus \cdots \oplus e_m$. Clearly $e_1 \oplus \cdots \oplus e_n$ does not operate as a left weak identity. In this paper if $e_i \in \mathcal{G},\ i \geq 2$, is placed in a joint receipt such that it is evaluated with $\succcurlyeq_i$ (so that $u_i(e_i) = 0$), then we call this e_i *a weak identity in the i-th year*.

Moreover, since $\mathcal{G}$ is assumed to be unbounded in $\succcurlyeq_i$, a certain type of unrestricted solvability is also necessary. But the description of the axiom is to be omitted.

For the problem of handling temporal discounting, the additive model of Eq. (2) gives the same solutions as Theorem 7.4 of Fishburn [4] or Theorem 6.15(i) of Krantz *et al* [2]. The description of these requires two notions [8]. $\succsim^n$ on $\mathcal{G}^n$ is *persistent* if and only if $x \oplus g_{\bar{i}} \succsim^n y \oplus g_{\bar{i}} \Leftrightarrow x \oplus g_{\bar{j}} \succsim^n y \oplus g_{\bar{j}}$ whenever $i,\ j \in \{1, \ldots, n\}$ and all four joint receipts are in $\mathcal{G}^n$. $\succsim^n$ on $\mathcal{G}^n$ is *impatient* if and only if $x \oplus \cdots \oplus x \succsim^n y \oplus \cdots \oplus y \Leftrightarrow g_1 \oplus \cdots \oplus g_{i-1} \oplus x \oplus y \oplus g_{i+2} \oplus \cdots \oplus g_n \succsim^n g_1 \oplus \cdots \oplus g_{i-1} \oplus y \oplus x \oplus g_{i+2} \oplus \cdots \oplus g_n$ whenever $i \in \{1, \ldots, n-1\}$ and the joint receipts are in $\mathcal{G}^n$.

Proposition 3: Assume that the additive representation of Eq. (2) holds. Then there exist an von Neumann-Morgenstern utility u on $\mathcal{G}$ and positive numbers $\pi_1, \ldots, \pi_n$ with $\sum_{i=1}^n \pi_i = 1$ such that, for all $g,\ h \in \mathcal{G}^n$,

$$g \succsim^n h \ \Leftrightarrow\ \sum_{i=1}^{n} \pi_i u(g_i) \geq \sum_{i=1}^{n} \pi_i u(h_i) \tag{3}$$

if and only if $\succsim^n$ is persistent. In particular if $\succsim^n$ is also impatient then $\pi_1 > \cdots > \pi_n$ in Eq. (3). Moreover, the π_i are unique for such u, and if a von Neumann-Morgenstern utility v satisfy this representation in place of u, then $v = \alpha u + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$.

IV. Decision Problems of Income Streams

A. Comparison between the models of Eqs. (2) and (3)

Example 4: Suppose that a software company plans new employment of engineers to cope with a rapid increase in demand. Since this industry is thriving, one can easily expect that many students and engineers in other companies will apply for the positions depending on working conditions. The company considers a salary system is very important for hiring competent engineers, and provides a choice between two sorts of two-year salaries. One guarantees an annual salary

$$x = (\$40000,\ 1.0)$$

for each of the first two years, and the other gives the combination of incentive wages,

$$\begin{aligned} g &= (\$37000, 0.5; \$43000, 0.5) \text{ for the first year,} \\ h &= (\$35000, 0.5; \$45000, 0.5) \text{ for the second year.} \end{aligned}$$

Which of the two-year salaries do applicants prefer?

These two sorts of two-year salaries are expressed as $x \oplus x$ and $g \oplus h$, respectively. We now consider preferences with the use of the additive models of Eqs. (2) and (3). Henceforth the occurrence of either $\pi_1 = 0$ or $\pi_2 = 0$ is allowed.

Assume first that a decision maker (an applicant) is risk averse in choosing a two-year salary. Then, by the definition,

$$u(g) < u(E(g)) \quad \text{and} \quad u(h) < u(E(h)).$$

Since $E(g) = x$ and $E(h) = x$, it follows that

$$\pi_1 u(g) + \pi_2 u(h) < \pi_1 u(x) + \pi_2 u(x)$$

no matter what value is given π_1 (π_2 is defined as $1 - \pi_1$). This implies that

$$g \oplus h \prec x \oplus x$$

is given by the additive model of Eq. (3). Assume conversely that the decision maker is risk seeking in choosing a two-year salary. To distinguish the present and the preceding utility, let u' denote a risk seeking type of utility. Then, by the definition,

$$u'(g) > u'(E(g)) \quad \text{and} \quad u'(h) > u'(E(h)).$$

Again since $E(g) = x$ and $E(h) = x$, we have

$$\pi_1 u'(g) + \pi_2 u'(h) > \pi_1 u'(x) + \pi_2 u'(x)$$

without respect to the value of π_1 (hence π_2), so that

$$g \oplus h \succ x \oplus x.$$

Thus the additive model of Eq. (3) gives a definite preference, $g \oplus h \prec x \oplus x$ or $g \oplus h \succ x \oplus x$, only according as the decision maker is risk averse or risk seeking.

We shall next consider the preferences explained by the additive model of Eq. (2). Assume that the decision maker is risk averse in a choice of salaries in the first year and risk seeking in a choice of salaries in the second year. Therefore his or her utilities in Eq. (2) are assumed to be defined by $u_1 = u$ and $u_2 = u'$. For simplicity, we now define $u(x) = 1.0$ and $u'(x) = 1.0$. The values of u and u' may be estimated at other outcomes as follows: with units in thousands of dollars,

$\times 10^3$ dollars	35	37	40	43	45
u	0.6	0.8	1.0	1.1	1.2
u'	0.4	0.6	1.0	1.5	2.0

Then since $u(g) = 0.95$ and $u'(h) = 1.2$ by Eq. (1), it follows that

$$U(g \oplus h) = \pi_1 u(g) + (1 - \pi_1) u'(h) = 1.2 - 0.25\pi_1.$$

Moreover,

$$U(x \oplus x) = (\pi_1 + (1 - \pi_1)) u(x) = 1.0.$$

From these two equations we deduce the following preferences

$$g \oplus h \ \begin{matrix} \succ \\ \sim \\ \prec \end{matrix} \ x \oplus x \quad \text{if } \pi_1 \begin{matrix} < \\ = \\ > \end{matrix} \frac{4}{5}.$$

It is, therefore, verified that if two sorts of utility functions are provided, then a variety of preferences occur depending on the values of π_1. In other words, the values of weights play a different role in explaining preferences than the shapes of utility functions do. Therefore the additive model of Eq. (2) will be superior in explanation ability to the additive model of Eq. (3).

We conclude this subsection with an additional comment. If even for the additive model of Eq. (3), there exist $g, h \in \mathcal{G}$ such that $u(g) > u(x) > u(h) > 0$ or $u(h) > u(x) > u(g) > 0$, then the above three types of preferences always occur according to the values of π_1. Of course, with the additive model of Eq. (2), a couple of concave (convex) utilities can also yield these preferences.

B. Comparison between one- and two-year salaries

Example 5: The company in Example 4 wishes to employ an engineer only if he or she is competent, but does not believe that such competence can be measured with complete accuracy. To solve this problem, the company adopts a policy hiring the necessary engineers and evaluating their ability during a two-year contract. It provides a choice between $x \oplus x$ and the following two-year salary: an annual salary

$$y = (\$43000,\ 1.0)$$

is given for the first year and options, extension of employment and resignation, for and after the second year. Applicants are informed that the prospect of their salaries for the second year will be

$$h' = (\$35000,\ 0.4;\ \$40000,\ 0.2;\ \$45000,\ 0.4),$$

each outcome of which is determined at the year-end. Which of the two-year salaries do applicants prefer?

Many applicants may estimate a weak identity e in the second year at (\$25000, 1.0) rather than at (\$0, 1.0), since they know \$25000 to be the least annual income among engineers occupied with this industry. The two-year salary newly offered by the company is expressed as $y \oplus h'$ or $y \oplus e$ according as extension or resignation is chosen. In what follows we consider preferences among $x \oplus x$, $y \oplus h'$, and $y \oplus e$ by using the additive model of Eq. (2).

Assume that a decision maker is risk averse in choosing a two-year salary. Then since $u(y) = 1.1$ and $u(h') = 0.92$ by Eq. (1),

$$U(y \oplus h') = \pi_1 u(y) + (1 - \pi_1)u(h') = 0.92 + 0.18\pi_1.$$

Also

$$U(x \oplus x) = 1.0.$$

From these two equations we obtain

$$y \oplus h' \;\begin{matrix} \succ \\ \sim \\ \prec \end{matrix}\; x \oplus x \quad \text{if } \pi_1 \gtreqless \frac{4}{9}.$$

Moreover, since $u(e) = 0$, the following holds:

$$x \oplus x \;\begin{matrix} \succ \\ \sim \\ \prec \end{matrix}\; y \oplus e \quad \text{if } \pi_1 \lesseqgtr \frac{10}{11}.$$

Note here that $y \oplus h' \succ y \oplus e$ always holds. It then turns out that the decision maker chooses $y \oplus h'$ if $\pi_1 > 4/9$ and $x \oplus x$ if $\pi_1 < 4/9$. Assume next that in choosing salaries the decision maker is risk averse in the first year and risk seeking in the second year. Since $u(y) = 1.1$ and $u'(h') = 1.16$ by Eq. (1), it follows that

$$\begin{aligned} U(y \oplus h') &= \pi_1 u(y) + (1 - \pi_1)u'(h') \\ &= 1.16 - 0.06\pi_1 \;>\; 1.0 \;=\; U(x \oplus x) \end{aligned}$$

for all $\pi_1 \leq 1$, so that $y \oplus h' \succ x \oplus x$ holds without respect to the value of π_1. Since $u'(e) = 0$, $y \oplus h' \succ y \oplus e$ always holds. Therefore $y \oplus h'$ turns out to be favorite. The company may expect an applicant to choose $y \oplus h'$ through these decision processes. Indeed, most companies will like an engineer who is properly confident of his or her own abilities. So they wishes an engineer who is willing to seek risk in the second year, or who does not overweight the second year even if he or she is averse to risk in both of the two years.

Finally, let us consider a preference structure of the decision maker who chooses $y \oplus e$. It may be suitable to assume that such a decision maker's right weak identity in the second year is $x =$ (\$40000, 1.0). Probably, he or she is very optimistic or is too much of a perfectionist to admit a decline in annual incomes. The prospect theory [1] reminds us of the fact that utility functions tend to be steeper for losses than for gains (see Fig. 2). For example, such a utility function u'' is given as

$$u''(35) = -1.0,\; u''(40) = 0.0,\; u''(45) = 0.5.$$

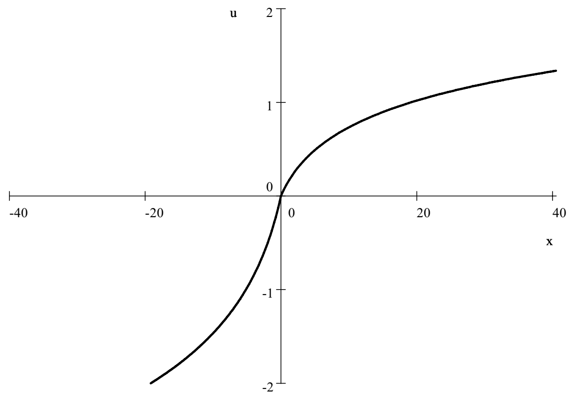

Fig. 2. A typical utility function in losses and in gains.

Then, by Eq. (1), $u''(h') = -0.2$. The utility of $y =$ (\$43000, 1.0) is greater than that of $x =$ (\$40000, 1.0) (i.e., $u(y) > u(x)$ or $u'(y) > u'(x)$) whether the decision maker is risk averse or risk seeking in the first year. Moreover, since $u''(e) = u''(x) = 0$ and $u''(h') < 0$, it follows that

$$U(y \oplus e) \geq U(x \oplus x) \quad \text{and} \quad U(y \oplus e) \geq U(y \oplus h').$$

Here the equality holds in the former inequality if $\pi_2 = 1$, and in the latter inequality if $\pi_1 = 1$. Therefore $y \oplus e$ is the favorite without respect to the value of π_1. This example informs us that the choice of a right weak identity is essential to the comparison between income streams with the distinct length of periods (i.e., between arbitrary joint receipts). Without doubt, the company is most unwilling to employ this sort of applicant.

V. Conclusion

This paper considered to apply the additive utility model of Eq. (2) to decision problems of income streams. First, it was verified that the values of weights played a different role than the shapes of utility functions did, in comparison with the additive model of Eq. (3). Secondly, by assuming the existence of an element acting like a right weak identity under the joint receipt operation, every one-year salary could be identified with a two-year salary. So we made it possible to consider choices between one- and two-year salaries.

Further research can be expected. The clarification of the way of estimating weights, along with the axiomatization for the additive model of Eq. (2), is an important study assignment (which is under study). It will be also worthwhile to examine the explanation ability of a weighted sum of rank-dependent utilities. More widely, temporal effects are very likely to appear in many decision problems, such as preference of brands in marketing, renewal of insurance, and animal choice behavior. It is meaningful to consider theoretically or empirically the possibility of applying weighted additive models associated with the concept of joint receipt to such decision problems.

Appendix

Proof: [Proposition 2] For the former assertion let $U(g) = \sum_{i=1}^{n} \pi_i u_i(g_i)$ and $U'(g) = \sum_{i=1}^{n} \pi'_i u_i(g_i)$, with $\pi'_i >$

$0,\ \sum_{i=1}^n \pi_i' = 1$, be additive representations of $\succsim^n$. Let $g_{(i,j)}$ denote an element $g_1 \oplus \cdots \oplus g_n$ with $g_k \in \mathcal{G}$ such that $g_k = e_k$ unless $k = i,\ j$. Since $u_i(e_i) = 0$ for all i, we have

$$\begin{aligned} U(g_{(i,j)}) &= \pi_i u_i(g_i) + \pi_j u_j(g_j), \\ U'(g_{(i,j)}) &= \pi_i' u_i(g_i) + \pi_j' u_j(g_j). \end{aligned}$$

Since U and U' preserve $\succsim^n$, it follows that

$$\frac{u_j(h_j) - u_j(g_j)}{u_i(g_i) - u_i(h_i)} \lesseqqgtr \frac{\pi_i}{\pi_j} \Leftrightarrow \frac{u_j(h_j) - u_j(g_j)}{u_i(g_i) - u_i(h_i)} \lesseqqgtr \frac{\pi_i'}{\pi_j'}$$

provided that $u_i(g_i) - u_i(h_i) \neq 0$. Note that each u_i is uniquely determined on $\mathcal{G}$ by the hypothesis, i.e., $u_i(f_i) = 1,\ u_i(e_i) = 0$, and further that $u_i(g_i), u_j(g_j),\ u_i(h_i)$, and $u_i(h_i)$ are variables in $\mathbb{R}$. So π_i/π_j must equal π_i'/π_j'. Since this is true for all distinct $i,\ j$, we see, in view of the assumption that $\sum_{i=1}^n \pi_i = \sum_{i=1}^n \pi_i' = 1$, that $\pi_i = \pi_i'$ for all i. Hence the π_i are unique for the u_i. To prove the latter assertion, let v_i be a von Neumann-Morgenstern utility scaled differently on $\langle \mathcal{G},\ \succcurlyeq_i \rangle$, so that $v_i = \alpha_i u_i + \beta_i$ for $\alpha_i > 0,\ \beta_i \in \mathbb{R}$, and let $V(g) = \sum_{i=1}^n \pi_i v_i(g_i)$. So $V(g) = \sum_{i=1}^n \pi_i(\alpha_i u_i(g_i)) + \beta$ where $\beta = \sum_{i=1}^n \beta_i$. If $\alpha_i = \alpha > 0$ for all i, then since $V(g) = \alpha U(g) + \beta$, it follows that $g \succsim^n h \Leftrightarrow V(g) \geq V(h)$. We show conversely that if V and U are additive representations of $\succsim^n$, then $\alpha_i = \alpha$ for all i. This is accomplished in a manner similar to that used in the proof of the former assertion. Substituting $g_{(i,j)}$ for g into U and V gives, respectively,

$$\begin{aligned} U(g_{(i,j)}) &= \pi_i u_i(g_i) + \pi_j u_j(g_j), \\ V(g_{(i,j)}) &= \alpha_i \pi_i u_i(g_i) + \alpha_j \pi_j u_j(g_j) + \beta. \end{aligned}$$

The order preservation of U and V implies that

$$\frac{u_j(h_j) - u_j(g_j)}{u_i(g_i) - u_i(h_i)} \lesseqqgtr \frac{\pi_i}{\pi_j} \Leftrightarrow \frac{u_j(h_j) - u_j(g_j)}{u_i(g_i) - u_i(h_i)} \lesseqqgtr \frac{\alpha_i}{\alpha_j} \cdot \frac{\pi_i}{\pi_j}$$

provided that $u_i(g_i) - u_i(h_i) \neq 0$. Since each u_i is uniquely determined and since $u_i(g_i),\ u_j(g_j),\ u_i(h_i)$, and $u_j(h_j)$ run over $\mathbb{R}$, α_i/α_j must equal 1, or $\alpha_i = \alpha_j$. This is true for all distinct $i,\ j$, and hence $\alpha_i = \alpha$ for all i. ■

Proof: [Proposition 3] To avoid confusion, let us replace π_i with ρ_i $(i = 1, \ldots, n)$ in Eq. (3). Assume that the representation of Eq. (2) reduces to $U(g) = \sum_{i=1}^n \rho_i u(g_i)$ for all $g \in \mathcal{G}^n$. Then since $\pi_i u_i = \rho_i u,\ \pi_j u_j = \rho_j u$ for $i,\ j \in \{1, \ldots, n\}$, we have $u_i = \{(\pi_j \rho_i)/(\pi_i \rho_j)\} u_j$, and so u_i and u_j are equivalent. Hence $x \succcurlyeq_i y \Leftrightarrow x \succcurlyeq_j y$ for all $x,\ y \in \mathcal{G}$. This along with A3 proves that $\succsim^n$ is persistent. Conversely, assume that $\succsim^n$ is persistent, i.e., that $x \oplus g_{\bar{i}} \succsim^n y \oplus g_{\bar{i}} \Leftrightarrow x \oplus g_{\bar{j}} \succsim^n y \oplus g_{\bar{j}}$. Substituting these four n-tuples into Eq. (2), we get $u_i(x) \geq u_i(y) \Leftrightarrow u_j(x) \geq u_j(y)$, so that u_i and u_j must be related by a positive affine transformation. Hence, for each i, $u_i(x) = \alpha_i u_1(x) + \beta_i$ holds for some $\alpha_i > 0,\ \beta_i \in \mathbb{R}$. Setting $u = u_1$, we have $U(g) = \sum_{i=1}^n (\alpha_i \pi_i) u(g_i) +$ constant. Let $\rho_i = (\alpha_i \pi_i)/\sum_{i=1}^n \alpha_i \pi_i$, so that $\rho_i > 0$ and $\sum_{i=1}^n \rho_i = 1$. In view of the fact that if U is order-preserving then the same is true for $U/\sum_{i=1}^n \alpha_i \pi_i$, we get $g \succsim^n h \Leftrightarrow \sum_{i=1}^n \rho_i u(g_i) > \sum_{i=1}^n \rho_i u(g_i)$ whenever $g,\ h \in \mathcal{G}^n$.

We consider the case where $\succsim^n$ is also impatient. From the unboundedness of $\mathcal{G}$ it is valid that $x \succ_i y$ for some $x,\ y \in \mathcal{G}$. Substitute the consequent inequality of impatience into Eq. (3) and arrange to get $(\pi_i - \pi_{i+1})(u(x) - u(y)) > 0$. Since $u(x) > u(y)$, we get $\pi_i > \pi_{i+1}$ for each $i = 1, \ldots, n-1$, implying that $\pi_1 > \cdots > \pi_n$. By the construction of ρ_i and of u, their uniqueness assertions follow from Proposition 2. ■

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