Fuzzy Ratings and Crisp Feedback in Fuzzy AHP for Supporting Human Decision Making

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Abstract—**People often rely on their feelings in choosing and deciding their behaviour in everyday life. Analytic Hierarchy Process (AHP) is one of the most popular tools for supporting human decision making, and several fuzzy extensions of AHP have been proposed. The present study revealed both psychological effects of both fuzzy ratings in fuzzy AHP and crisp presentations of the results from fuzzy AHP. That is, fuzzy ratings in fuzzy AHP could incorporate the fuzziness of a person's feelings in his/her decision making, and crisp feedback could help him/her make his/her decision, especially when a decision maker is being puzzled about his/her choice. In this study, the deffuzification was conducted by Thurstone's paired comparison method, which can exaggerate the priority of only one specific alternative with respect to some characteristic.**

Ⅰ. INTRODUCTION

People often rely on their feelings in choosing and deciding their behaviour in everyday life. Therefore, an appropriate supporting tool for people to reach satisfying goals is very important[1,2], and one of the most popular tools for supporting human decision making is Analytic Hierarchy Process (AHP) by Saaty[3,4]. In Saaty's AHP, a person is asked to supply ratios for each pairwise comparison between alternatives $A_1, A_2, \dots, A_m$ and also between the criteria $C_1, C_2, \dots, C_n$. However, it seems to be more natural for people to feel that the ratio is approximately 5 to 1 or that the ratio is between 4 to 1 and 6 to 1 rather than for people to feel that the ratio is exactly 5 to 1.

Therefore, several extensions of Saaty's AHP[5,6] have been proposed by using fuzzy theory[7] which provides a mathematical method able to deal with feelings and cognitive processes which are too imprecise to be dealt with by classical mathematical techniques. Fuzzy AHPs have used fuzzy ratings, which can incorporate the fuzziness and vagueness of a person's feelings in his/her decision making[8]. However, the indeterminancy about the final fuzzy weights might make it more difficult for the decision maker to understand the suggestion of the results and to choose a specific alternative. It seems that the decision maker often wants the result which exaggerates the ascendancy of only one alternative.

Therefore, in order to defuzzificate the fuzzy weights from fuzzy AHP and to exaggerate the superiority of only one alternative, the present study tries to apply Thurstone's paired comparison method[9,10] to the results from fuzzy AHP. That is, we propose a method in which people can represent their fuzziness and vagueness in their feelings by fuzzy ratings and can get the results of AHP as the crisp numbers. Moreover, the present study provides a new way of looking at the effective use of fuzzy AHP for supporting human decision making.

Ⅱ. AHP AND PAIRED COMPARISONMETHOD

A. Saaty's AHP

Suppose we wish to compare a set of n alternatives in pairs according to their relative weights which are assumed to belong to a ratio scale. Denote the alternatives by $A_1, A_2, \dots, A_m$ and their weights by $w_1, w_2, \dots, w_m$. The pairwise comparisons may be represented by a matrix as follows:

$$\boldsymbol{A} = [\, w_i / w_j \,], \quad i, j = 1, 2, \dots, m. \qquad (1)$$

Since this matrix has positive entries everywhere and satisfies the reciprocal properties, it is called a positive reciprocal matrix. If we multiply this matrix by the transpose of the vector $\boldsymbol{w'}=[w_1, w_2, \dots, w_m]$, we obtain the vector $m\boldsymbol{w}$.

$\boldsymbol{Aw}=m\boldsymbol{w}$. (2)

If we only have $\boldsymbol{A}$ and want to estimate $\boldsymbol{w}$, we have to solve the equation $(\boldsymbol{A} - m\boldsymbol{I})\boldsymbol{w}=\boldsymbol{0}$ in the unknown $\boldsymbol{w}$. This has a nonzero solution if and only if m is an eigenvalue of $\boldsymbol{A}$. Since every row of $\boldsymbol{A}$ is a constant multiple of the first row, $\boldsymbol{A}$ has unit rank. That is, we can get the vector $\boldsymbol{w}$ as the eigenvector of $\boldsymbol{A}$.

Saaty's AHP assumes a hierarchical structure. For each criterion C_k in a hierarchy, a person is asked to supply ratios a_{ij} for each pairwise comparison between alternatives A_1, A_2, …, A_m. The ratio a_{ij} indicates the strength with which A_i dominates A_j for this person. The numbers for the ratios are usually taken from the set { 1, 2, …, 9}. In Saaty's scale, 1, 3, 5, 7, and 9 mean "equally importance," "weak importance of one over another," "essential or strong importance," "demonstrated importance," and "absolute importance," respectively. 2, 4, 6, and 8 mean intermediate values between the two adjacent judgments. If a_{ij} is equal to 5/1, then a_{ji} is taken as 1/5. That is, $a_{ji}=a_{ij}^{-1}$ and $a_{ii}=1$ for all i. Then we obtain a positive reciprocal matrix $\boldsymbol{A}_{(k)}$ for each criterion C_k.

The matrix $\boldsymbol{A}_{(k)}$ obtained from a person's paired comparisons dose not satisfy the property $a_{ij}a_{jk}=a_{ik}$, and $\boldsymbol{A}_{(k)}$ usually has m eigenvalues and eigenvectors. Therefore, let λ_{max} be the largest eigenvalue of $\boldsymbol{A}_{(k)}$ and $\boldsymbol{w}_{(k)}'=[w_{1k}, w_{2k}, \dots, w_{mk}]$ the associated normalized eigenvector. Each element of the eigenvector $\boldsymbol{w}_{(k)}$, $k=1,2,...,n$, is the estimate of the weight of each alternative for each criterion.

We also obtain a positive reciprocal matrix $\boldsymbol{C}$ for the pairwise comparisons of the criteria and get the estimates of the weights as the elements of the normalized eigenvector $\boldsymbol{c}'=[c_1, c_2, \dots, c_n]$. The final weight for alternative A_j is

$$f_j= w_{j1}\times c_1 +w_{j2}\times c_2 + \cdots + w_{jn}\times c_n. \quad (3)$$

B. Fuzzy AHP

We use the fuzzy AHP by Buckley[5] for its simple treatment, that is, it need only geometric means for computing fuzzy weights. In Buckley's fuzzy AHP, the numbers of ratios are fuzzy numbers. As shown in Figure 1, the type of fuzzy numbers used by Buckley's fuzzy AHP is described by $(m_1/m_2,\ m_3/m_4)$ where $0 < m_1 \leqq m_2 \leqq m_3 \leqq m_4$.

Consider two trapezoidal fuzzy numbers $\overline{M}=(m_1/m_2,\ m_3/m_4)$ and $\overline{N}=(n_1/n_2,\ n_3/n_4)$. For addition, subtraction, multiplication, division, and inverse, the following approximation formula are proposed:

$$\overline{M}+\overline{N}=(m_1+n_1 / m_2+n_2,\ m_3+n_3/m_4+n_4) \quad (4)$$

$$\overline{M}-\overline{N}=(m_1-n_4 / m_2-n_3,\ m_3-n_2/m_4-n_1) \quad (5)$$

$$\overline{M}\times\overline{N}\approx(m_1n_1\wedge m_1n_4\wedge m_4n_1\wedge m_4n_4 \ / m_2n_2\wedge m_2n_3\wedge m_3n_2\wedge m_3n_3,\ m_2n_2\vee m_2n_3\vee m_3n_2\vee m_3n_3 \ / m_1n_1\vee m_1n_4\vee m_4n_1\vee m_4n_4) \quad (6)$$

$$\overline{M}\div\overline{N}\approx(m_1/n_1\wedge m_1/n_4\wedge m_4/n_1\wedge m_4/n_4 \ / \ m_2/n_2\wedge m_2/n_3\wedge m_3/n_2\wedge m_3/n_3,\ m_2/n_2\vee m_2/n_3\vee m_3/n_2\vee m_3/n_3 \ / \ m_1/n_1\vee m_1/n_4\vee m_4/n_1\vee m_4/n_4) \quad (7)$$

$$M^{-1}\approx(m_4^{-1}/m_3^{-1},\ m_2^{-1}/m_1^{-1}). \quad (8)$$

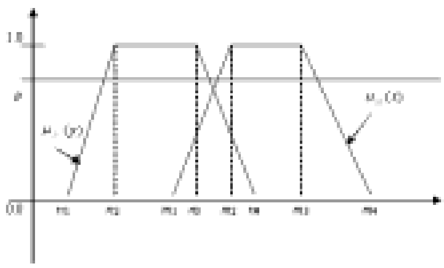

Figure 1 An example of membership functions of fuzzy numbers

If a person believes that alternative A_i is more important than A_j, then the fuzzy ratio $\overline{a}_{ij}=(\alpha_{ij}/\beta_{ij},\ \gamma_{ij}/\delta_{ij})$ has $\alpha,\ \beta,\ \gamma,\ \delta\in\{1, 2, \dots, 9\}$ and $\overline{a}_{ji}$ is taken as $\overline{a}_{ij}^{-1}=(\delta_{ij}^{-1}/\gamma_{ij}^{-1},\ \beta_{ij}^{-1}/\alpha_{ij}^{-1})$. Let $\overline{A}$ be the $m\times m$ fuzzy matrix of all paired comparisons for the alternatives A_1, A_2, …, A_m. The elements in $\overline{A}$ are $\overline{a}_{ij}$ where $\overline{a}_{ij}=\overline{a}_{ji}^{-1}$ and $\overline{a}_{ii}=(1/1,1/1)$ for all i. $\overline{A}$ is called a fuzzy positive reciprocal matrix.

In fuzzy AHP, if we obtain a fuzzy positive reciprocal matrix $\overline{A}_{(k)}$ of pairwise comparisons for each criterion C_k and a fuzzy positive reciprocal matrix $\overline{C}$ for the pairwise comparisons of the criteria, then we can obtain the final fuzzy weight of the alternatives A_j by the equation:

$$\bar{f}_j=(\overline{w}_{j1}\odot\bar{c}_1)\oplus(\overline{w}_{j2}\odot\bar{c}_2)\oplus\cdots\oplus(\overline{w}_{jk}\odot\bar{c}_k), \quad (9)$$

where $\oplus$ and $\odot$ mean the standard fuzzy addition and multiplication.

However, since Saaty's largest eigenvalue procedure for determining the weights is not readily extended to fuzzy matrices, Buckley[5] proposed the following method. Now consider a fuzzy positive reciprocal matrix $\overline{A}=[\overline{a}_{ij}]$ where $\overline{a}_{ij}=(\alpha_{ij}/\beta_{ij},\gamma_{ij}/\delta_{ij})$. We will determine the membership function μ_i for w_i. Let

$$fi(y)=[\prod_{j=1}^{m}((\beta_{ij}-\alpha_{ij})y+\alpha_{ij})]^{1/m} \quad (10)$$

$$g_i(y)=[\prod_{j=1}^{m}((\gamma_{ij}-\delta_{ij})y+\delta_{ij})]^{1/m} \quad (11)$$

for $0\leqq y\leqq 1$. Define

$$\alpha_i=[\prod_{j=1}^{m}\alpha_{ij}]^{1/m} \quad (12)$$

and

$$\alpha=\sum_{i=1}^{m}\alpha_i . \quad (13)$$

Similarly, define β_i, β, γ_i, γ, δ_i and δ. Finally, let

$$f(y)=\sum_{i=1}^{m}fi(y) \quad (14)$$

$$g(y)=\sum_{i=1}^{m}g_i(y) . \quad (15)$$

The fuzzy weights $\overline{w}_i$ are determined by $(\alpha_i\delta^{-1}/\beta_i\gamma^{-1},\gamma_i\beta^{-1}/\delta_i\alpha^{-1})$ where the graph of μ_i is zero to the left of $\alpha_i\delta^{-1}$, $x=f_i(y)/g(y)$ on the interval $[\alpha_i\delta^{-1},\ \beta_i\gamma^{-1}]$, a horizontal line from $(\beta_i\gamma^{-1},\ 1)$ to $(\gamma_i\beta^{-1},\ 1)$, $x=g_i(y)/f(y)$ on the interval $[\gamma_i\beta^{-1},\delta_i\alpha^{-1}]$, and zero to the right of $\delta_i\alpha^{-1}$.

C. Paired comparison method

Paired comparison method by Thurstone[9,10] is as follows. A person's sensations X_i (i=1, 2, … , n) is a normal distribution N(S_i, σ^2). The probability of times X_i exceeds X_j is

$$p_{ij}=P(X_i>X_j)=P(X_i-X_j>0)$$

$$=\frac{1}{\sqrt{2\pi}\sigma_{ij}}\int_0^{\infty}\exp[-\frac{\{d_{ij}-(S_i-S_j)\}^2}{2\sigma_{ij}^{\ 2}}]dd_{ij} \quad (16)$$

where $d_{ij}=X_i-X_j$ and $\sigma_{ij}^{\ 2}=2\sigma^2(1-\rho)$.

Assigning the scale factor so that

$$\sigma_{ij}^{\ 2}=2\sigma^2(1-\rho)=1 , \quad (17)$$

we can rewrite the above equation

$$p_{ij}=\frac{1}{\sqrt{2\pi}}\int_{S_j-S_i}^{\infty}\exp\{-\frac{x}{2}\}dx . \quad (18)$$

When we use the observed proportions p_{ij}, we can get S_j-S_i from the normal distribution table.

Ⅲ. EXPERIMENT

An experiment was conducted to confirm both the effects of fuzzy ratings in the fuzzy AHP and the crisp presentation showing the results of the fuzzy AHP.

A. Method

The subjects were 5 male graduate students and 1 female undergraduate student. On the assumption that each subject had to purchase one of the three kinds of cellular phones, A_1, A_2, and A_3, he/she was asked to answer both AHP and the fuzzy AHP which have the hierarchical structure as shown in Figure 2. The subjects replied a number from the set {1, 2, ⋯, 9} for AHP. For the fuzzy AHP, they replied four numbers which corresponded to a fuzzy number as shown in Figure 3. A_1, A_2, and A_3 were products of "Docomo," "au," and "J-phone."

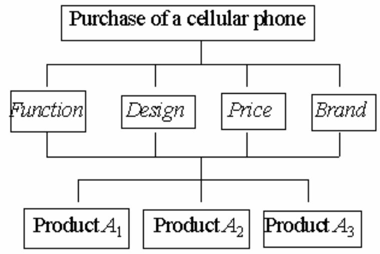

Figure 2 Hierarchy for AHP

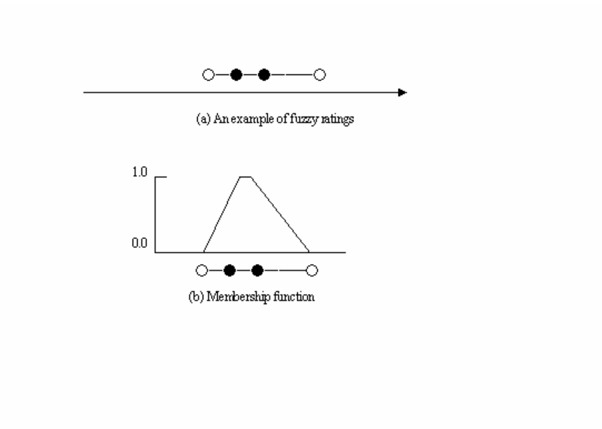

Figure 3 An example of fuzzy ratings and its membership function in the fuzzy AHP.

After pairwise comparisons for both AHP and the fuzzy AHP, they were required to ask the following 5 questions: 1) Which rating scale is easier for you to reply?; 2) Which rating scale can express your feeling better?; 3) Which rating method are you more familiar with?; 4) Which rating scale takes more time for you to reply?; 5) Which rating scale do you think you more likely use when you compare two things with respect to some characteristic in everyday life?

We made three kinds of presentation forms that presented the results of the fuzzy AHP: 1) fuzzy presentation form which showed the final fuzzy weights of the fuzzy AHP, 2) ranking presentation form which indicated the ranking obtained by Buckley's method[5], and 3) crisp presentation form obtained by Thurstone's paired comparison method which can defuzzificate the fuzzy weights and exaggerate the priority of one specific alternative.

Ranking of two fuzzy numbers was oonducted according to the following method proposed by Buckley. Let $\overline{M}$ and $\overline{N}$ be two fuzzy numbers with membership functions μ_M and μ_N, as shown in Figure 1. Define

$$v(\overline{M} \geq \overline{N}) = \sup_{x \geq y}(\min(\mu_M(x), \mu_N(y))). \tag{19}$$

It is assumed that $\overline{M}$ is greater than $\overline{N}$, written $\overline{M} > \overline{N}$, if $v(\overline{M} \geq \overline{N}) = 1$ and $v(\overline{N} \geq \overline{M}) < \theta$, where θ is some fixed positive fraction less than or equal to one. Moreover, if

$$\min(v(\overline{M} \geq \overline{N}), v(\overline{N} \geq \overline{M})) \geq \theta \tag{20}$$

then $\overline{M} \approx \overline{N}$. In this study, we set $\theta = 0.7$.

The crisp presentation was obtained as follows. We apply the paired comparison method by Thurstone[9,10] to all the paired comparisons of the numbers each of which consists of the fuzzy number for A_i, i=1,2,3.

The subjects were asked to compare the three presentation forms and to answer the following questions: 1) If you are puzzled about which product you should purchase, which presentation form do you think can help you make your decision? Please rank the three forms with respect to the effectiveness for supporting your decision making.; 2) Please rank the three forms with respect to the goodness of fit to your feelings when purchasing a product in our everyday life.; 3)Please rank the three forms with respect to the familiarity.

B. Results and discussion

(1) Comparisons of ratings

The results suggest that although the crisp ratings used by AHP is easier to reply, more familiar, and takes less time to reply, the fuzzy ratings used by the fuzzy AHP can model our daily response mode. That is, the subjects felt that the fuzzy ratings could express their feelings better, and the subjects felt that they more likely use the fuzzy rating scale when they compared two things with respect to some characteristics in everyday life. It seems that the fuzzy rating method can provide a response mode that allows for fuzziness.

(2) Comparisons of presentation forms

For example, the fuzzy weights of the criteria for Subject 1 were (0.067/0.108, 0.111/0.228), (0.030/0.043, 0.044/0.084), (0.203/0.294, 0.313/0.474), and (0.261/0.528, 0.561/0.996) for *function*, *design*, *price*, and *brand*, respectively. Moreover, the final fuzzy weights for A_1, A_2, and A_3 were (0.165/0.500, 0.529/1.477), (0.153/0.325, 0.345/0.772), and (0.065/0.147, 0.155/0.403), respectively. The ranking by Buckley's method was $A_3 < A_2 \approx A_1$. The scores obtained from the paired comparison method by

Thurstone were 0.67, 0.053, -0.73 for A_1, A_2, and A_3, respectively. The paired comparison results exaggerated.

The results from the questions indicate the superiority of the crisp presentation about both "effectiveness of supporting human decision making" and "goodness of fit to human feelings."

Ⅳ.CONCLUSION

In spite of the superiority of the fuzzy ratings for the fuzzy AHP, the presentation of the results of the fuzzy AHP was not effective for a decision maker to choose an alternative. That is perhaps because it is difficult for a person to understand the meaning of fuzzy numbers or membership functions, and because the fuzzy results might increase his/her hesitation about which alternative he/she should decide to choose.

Our experiments revealed that exaggerating the priority of only one specific alternative with respect to some characteristic could help a decision maker make his/her decision. This result agrees with Matsuda, Yamashita, and Tamura[11]. Especially, when a decision maker is deeply puzzled about which alternative should be chosen, it seems that our presentation method seems to be very effective for supporting his/her decision making. That is, our research can provide a new way of looking at the effective use of the fuzzy AHP for supporting human decision making.

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generally exaggerate the superiority of only one alternative. In this case, the superiority of A_1 was

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