A Method for Computing a Global Optimal Solution of Continuous Optimization Problems

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Abstract - **This paper proposes a systematic and efficient method for finding multiple local optimal solutions or a global optimal solution on nonlinear optimization problems. The method constructs a gradient vector field associated with an objective function and searches several local optimal solutions via saddle points on the ridge of the objective function. Identifying saddle points must be a challenging task for this research. The saddle points are also referred to as type I unstable equilibrium points (UEPs) or decomposition points (DPs). In this paper, we apply a bifurcation and a continuation method of nonlinear dynamics to finding the DPs. The continuation method traces a path of local optimal solutions (stable equilibrium points) and related DPs continuously and efficiently. This method has ability to find multiple local optimal solutions or a global optimal solution with reasonable computational time.**

I. INTRODUCTION

A method for obtaining a global optimal solution for general nonlinear optimization problems is highly significant technology to realize energy savings and cost reductions in industrial problems. Researchers have presented a lot of methods for searching the global optimal solution, for example, random multi-start local search, dynamic tunneling algorithm, genetic algorithm (GA), simulated annealing (SA), tabu search (TS), and particle swarm optimization (PSO) [1][2][3]. Some of the methods are referred to as metaheuristics and they might provide a plausible optimal solution with required computational time. However, these conventional methods are still insufficient, particularly for practical large-scale problems. Our method will provide a new idea to overcome this issue.

A new systematic method based on the concepts of stability regions [4][5] and quasi-stability regions [6][7][8] of nonlinear autonomous dynamical systems for searching multiple local optimal solutions has been presented in [9]. This method can search the multiple local optimal solutions or a global optimal solution systematically and robustly. Nevertheless, they have a problem of computational time. Our proposed method overcomes this time-consuming issue and presents a novel systematic and efficient method.

The concrete procedure for searching the global optimal solution includes the following steps.

Step 1: Construct a gradient vector field associated with the given objective function.

Step 2: Approach one of the local optimal solutions from any initial condition by a certain local search method.

Step 3: Escape from a convergence region of the obtained local optimal solution of the Step 2 through a decomposition point. Here, the decomposition point is the saddle point that has a one-dimensional unstable manifold. The number of the unstable manifold of the decomposition point is calculated by eigenvalues and eigenvectors.

Step 4: Approach another local optimal solution by moving along the opposite direction of the unstable manifold of the decomposition point. Go back to Step 3.

Step 5: Choose a local optimal solution that has a lowest value of the objective function as a global optimal solution.

The challenging task of this method is to identify the locations of the decomposition points on the quasi-stability boundary of the obtained local optimal solution (stable equilibrium point). The method proposed in [9] integrates the vector filed from many initial conditions. These initial conditions are set the proximity of the obtained local optimal solutions. No indices exist for setting the initial conditions and identifying the decomposition points is a time-consuming procedure, particularly, for large-scale systems. Therefore, identifying the decomposition points with reasonable computational time has been required.

This paper presents a novel approach to identify the locations of the decomposition points and proposes a systematic and efficient method for obtaining the global optimal solution. This proposed method uses an idea of the parameter dependence of the vector filed to overcome the time-consuming issue. The parameter dependence is called a bifurcation phenomenon [10]-[14]. In general, the vector field has parameter dependence and the locations of the local optimal solutions and the decomposition points must be changed with slow parameter variations. In other words, local optimal solutions and the decomposition points approach, and coalesce at a bifurcation point. Therefore, if we trace paths of the local optimal solutions and the decomposition points continuously, we will specify the locations of the decomposition points from the obtained local optimal solutions. The numerical computation of the tracing the paths can be done by a numerical continuation method [15][16][17]. The continuation method is a fast numerical algorithm to trace the paths of the local optimal solutions and the decomposition points. Besides no integration from many initial conditions is required for identifying the locations of the decomposition points. Consequently, we can find the local optimal solutions and the decomposition points one after another by the combination of the certain local search method and the numerical continuation method, and obtain the global optimal solution with expected computational time.

We verified our proposed method by the 2-dimensional Hump Camel-Back function [9] and the 2-dimensional Griewank function [18]. The method proposed in [9] required at least 20-time integration for finding six local optimal solutions of the 2-dimensional Hump Camel-Back function, while our proposed method can find these solutions with only one path tracing by numerical continuation method. Obviously our proposed method computes the solutions faster than the method in [9]. Furthermore, the proposed method found the global optimal solutions of the 2-dimensional Griewank function with expected computational time. This proposed method overcomes the time-consuming issue of the previous method. The numerical results show the effectiveness of our proposed method. The proposed method has ability to search the global optimal solution with practical computational time.

II. PROBLEM FORMULATION

Consider the following unconstrained global optimization problem

$$\min\{c(x) : x \in \Re^n\} \quad \cdots\cdots \quad (1)$$

where $c \in C^2$. The function $c(x)$ is assumed to be bounded below so that its global minimal solution exists and the number of local minimal solutions are finite.

We can transform the global optimization problems into the problems of nonlinear dynamics. Therefore, we are able to construct the following gradient vector filed (2) from the objective function $c(x)$, and we can find the global optimal solution in the gradient vector filed.

$$\dot{x} = -\nabla c(x) \quad \cdots\cdots \quad (2)$$

III. QUASI-GRADIENT SYSTEMS

We briefly introduce some concepts that play a central role in the theory of nonlinear dynamics [4][10]-[14].

We consider a nonlinear dynamical system described by

$$\dot{x}(t) = f(x(t)) \quad \cdots\cdots \quad (3)$$

where the state vector $x(t)$ of this dynamical system belongs to the Euclidean space $\Re^n$, and the function $f : \Re^n \to \Re^n$ satisfies the existence and uniqueness theorem. A solution curve of (3) starting from x at t=0 is called a trajectory, denoted $\Phi(x,\cdot) : \Re \to \Re^n$.

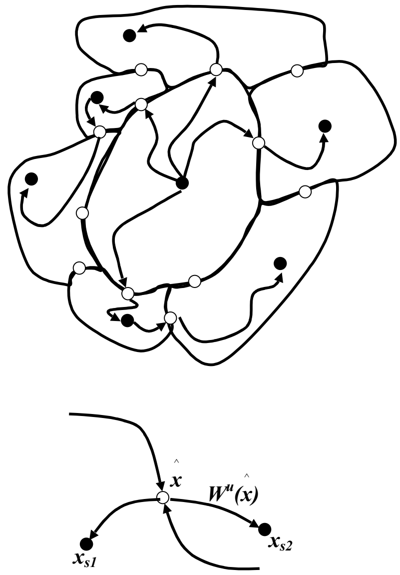


Figure 1 A general structure of the gradient vector filed (above) and the connection of equilibrium points (bottom).

A state vector x^* is called an *equilibrium point* of (3) if $f(x^*)$=0. A state vector x is called a *regular point* if it is not an equilibrium point. E denotes a set of equilibrium points. We say that an equilibrium point of (3) is *hyperbolic* if the Jacobian of $f(.)$ at x^*, denoted $J_f(x^*)$, has no eigenvalues with a zero real part. A hyperbolic equilibrium point is a (asymptotically) *stable equilibrium point* if all the eigenvalues of its corresponding Jacobian have negative real parts; otherwise it is an unstable equilibrium point. If the Jacobian of the equilibrium point x^* has exactly k eigenvalues with positive real parts, then it is called a type-k equilibrium point. If k=0, then the equilibrium point is called a stable equilibrium point (SEP), or a *sink* (or attractor). If k=n, then the equilibrium point is called an unstable equilibrium point (UEP), or a *source* (or repeller).

Let x^* be a hyperbolic equilibrium point. Its *stable and unstable manifolds* $W^s(x^*)$, $W^u(x^*)$are defined as follows:

$$W^s(x^*) := \left\{x \in \Re^n : \lim_{t\to\infty} \Phi(x,t) = x^*\right\} \quad \cdots\cdots \quad (4.1)$$

$$W^u(x^*) := \left\{x \in \Re^n : \lim_{t\to-\infty} \Phi(x,t) = x^*\right\} \quad \cdots\cdots \quad (4.2)$$

A useful concept for a system that has multiple stable equilibrium points is a *stability region* (or domain of attraction). The stability region of a stable equilibrium point x_s is defined as

$$A(x_s) := \left\{x \in \Re^n : \lim_{t\to\infty} \Phi(x,t) = x_s\right\} \quad \cdots\cdots \quad (5)$$

From a topological point of view, the stability region is an open, invariant, and connected set. The boundary of stability region is called a *stability boundary* (or separatrix) of x_s and is denoted $\partial A(x_s)$. The stability boundary is topologically a (n-1)-dimensional closed invariant set. A concept of Quasi-stability region or practical stability region plays important role in this paper. The practical stability region of a stable equilibrium point x_s, denoted $A_p(x_s)$, is the open set $int\,\overline{A}(x_s)$, where $\overline{A}$ denotes the closure of A and $int\,\overline{A}$ denotes the interior of $\overline{A}$. The difference between the stability region (boundary) and the practical stability region (boundary) has been investigated in [4]-[8] in detail.

Figure 1 illustrates the general structure of the quasi-gradient vector filed. This vector field possesses multiple equilibrium points. The black and white dots indicate SEPs (local optimal solutions) and UEPs (decomposition points) respectively. The quasi-stability regions surround the SEPs. The decomposition points are on the quasi-stability boundaries. According to the proposition 1, all SEPs are connected through the unstable manifold $W^u(\hat{x})$ of the decomposition points (see, Figure 1). Therefore, if we identify decomposition points efficiently, we can find local optimal solutions, which are candidates of a global optimal solution, with reasonable computational time.

Proposition 1 (Decomposition points and the stable equilibrium points) [9]

Let x_s^1 be a stable equilibrium point of quasi-gradient system (2) and $\hat{x}$ be a decomposition point on the practical stability boundary $\partial A(x_s^1)$. If every equilibrium point of the quail-gradient system (2) lying on $\partial A(x_s^1)$ is hyperbolic and its stable and unstable manifolds satisfy the transversal condition, then there exists another equilibrium point x_s^2 to which the 1-D unstable manifold of $\hat{x}$ converges.

IV. A SYSTEMATIC METHOD

As studied in the previous sections, a systematic and efficient method for finding several local optimal solutions includes the following steps:

Step 1: Construct a gradient vector field from the objective function $c(x)$.

Step 2: Integrate the constructed gradient vector field from any initial condition. Identify the first local optimal solution.

Step 3: Find all decomposition points on the quasi-stability boundary of the obtained local optimal solution.

Step 4: Locate the 1-D unstable manifold of the found decomposition points, and find another local optimal solution. In this step we clarify the connection of the decomposition points and the local optimal solutions. Repeat Step 3&4, and obtain multiple local optimal solutions.

Step 5: Compare all the obtained local optimal solutions and choose a global optimal solution.

V. IDENYIFYING DECOMPOSITION POINTS

A. PREVIOUS METHOD [9]

The previous method is to construct a reflected gradient that has a trajectory that starts from a stable equilibrium point and converges to a decomposition point. This method is required to integrate the reflected gradient from many initial conditions to find several decomposition points. Identifying the decomposition points depend on initial values (see, Figure 2). Therefore, this previous method has a time-consuming issue, particularly for large-scale problems.

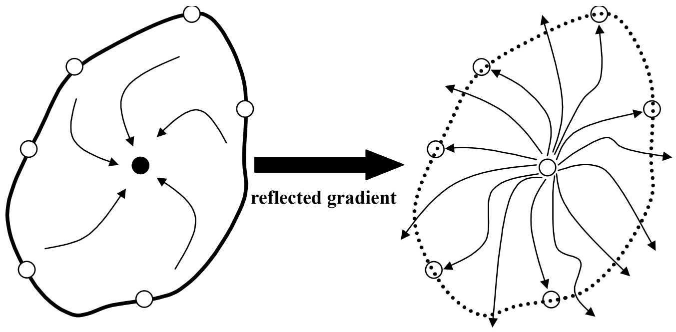


Figure 2 The previous method [9] for finding decomposition points.

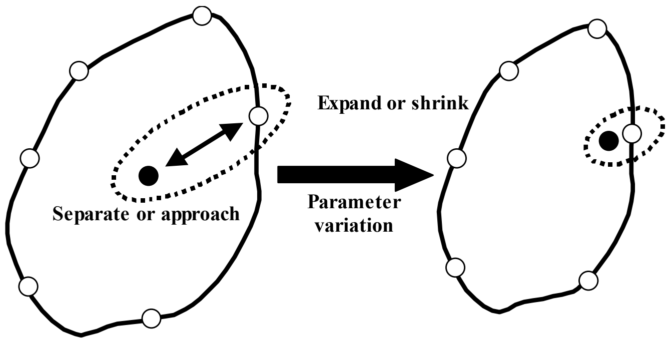


Figure 3 The proposed method for finding decomposition points.

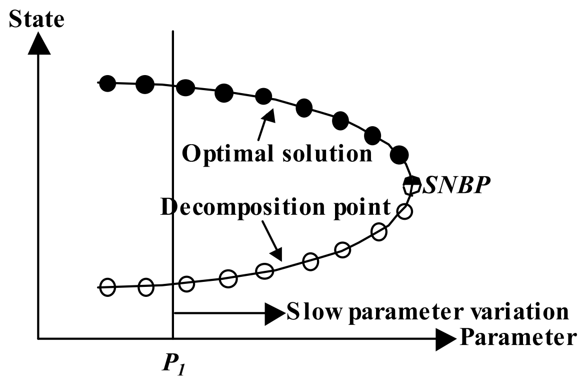


Figure 4 The paths of the optimal solutions and the decomposition points near the bifurcation point.

B. PROPOSED METHOD

The new idea comes from a bifurcation phenomenon of equilibrium points in the vector field. In general a vector field has parameter dependence. Equilibrium points must separate or approach each other, and shapes of quasi-stability regions must expand or shrink with slow parameter variations (See, Figure 3).

The (saddle-node) bifurcation is the mechanisms by which equilibrium points are created and destroyed (see, Figure 4). As a parameter is varied, two equilibrium points move toward each other, and mutually annihilate at saddle-node bifurcation point (SNBP). The approach and collision of equilibrium points identify the location of all the equilibrium points. After the SNBP, we can clarify a bottom path (decomposition points). Therefore, if we back the parameter to P_1, we identify the decomposition point at P_1.

A continuation method traces equilibrium points of the vector fields continuously. Therefore, once we find a stable equilibrium point, the found stable equilibrium point finds unstable equilibrium points one after another. Finally, we compare the values of the all obtained local optimal solutions, and then we choose one of the local optimal solutions as a global optimal solution.

VI. NUMERICAL STUDY

For the purpose of expressing the effectiveness of the proposed method, we study the following 2-dimensional Hump Camel-back function and 2-dimensional Griewank function.

A. 2-DIMENSIONAL HUMP CAMEL-BACK FUNCTION

Minimize the 2-dimensional Hump Camel-back function [9] described by

$$c(x_1, x_2) = ax_1^2 + bx_1^4 + cx_1^6 - fx_1x_2 + dx_2^2 + ex_2^4 \quad \text{(6)}$$

where, a=4, b=-2.1, c=1/3, d=-4, e=4, f=1.

The 2-dimensinal Hump Camel-Back function has 6 optimal solutions and two of them are the global optimal solutions.

Step 1: Construct the quasi-vector field

The quasi-vector field and the Jacobian matrix of the function (6) are follows.

Quasi-vector field:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2ax_1 - 4bx_1^3 - 6cx_1^5 + fx_2 \\ fx_1 - 2dx_2 - 4ex_2^3 \end{pmatrix} \quad \text{(7)}$$

Jacobian matrix J:

$$J = \begin{pmatrix} -2a - 12bx_1^2 - 30cx_1^4 & f \\ f & -2d - 12ex_2^2 \end{pmatrix} \quad \text{(8)}$$

Step 2: Obtain the first local optimal solution

Integrate the constructed quasi-vector field (7) from the initial condition (0.5, 0.5), and obtain the first local optimal solution (0.0898, 0.7127).

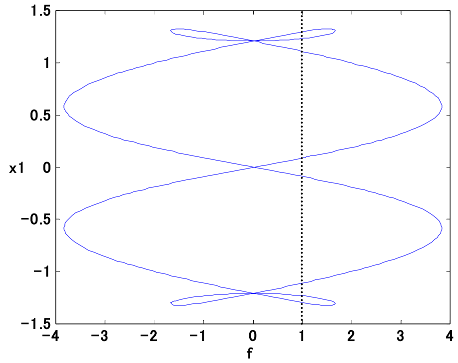
Step 3: Find decomposition points

Trace equilibrium points from the initial condition (0.0898, 0.7127), which is obtained as the local optimal solution in the Step 2, using Continuation method. Figure 5 shows an example of the bifurcation diagram when the parameter is f. The example shows that 7 equilibrium points are found from the intersection between the branches (paths) and f=1. The stability of the found equilibrium points is estimated by the eigenvalues of the Jacobian matrix (8). Follows are results.

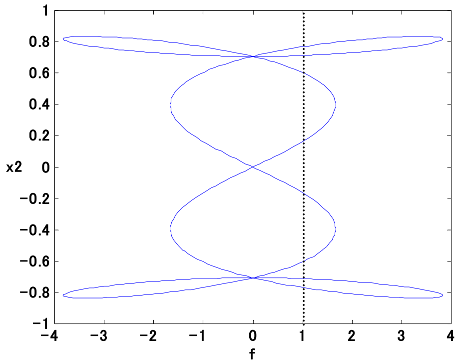
Decomposition points: (1.1092, 0.7683), (1.2961, -0.6051), (-1.1092, -0.7683), (-1.2961, 0.6051).

Local optimal solution: (-0.0898, -0.7127).

Type II unstable equilibrium points: (1.2302, -0.1623), (-1.2302, 0.1623). We can obtain other equilibrium points on the quasi-stability boundary with the other parameter variation using the continuation method.



(a) f versus x_1.



(b) f versus x_2.

Figure 5 The bifurcation diagrams of the 2-dimenisional Hump Camel-Back function when the parameter is f.

Note: We can obtain not only unstable equilibrium points but also stable equilibrium points by continuation method.

Step 4: Obtain the second, third, ...optimal solutions

Integrate the vector field from the intersection between the normalized unstable eigenvector of the decomposition points and the ε-ball [9]. Table 1 shows the obtained decomposition points (DPs) and the local optimal solutions and their connection.

Step 5: Identify the global optimal solution

We obtained the 6 local optimal solutions by above steps. Table 2 shows the obtained local optimal solutions and their values of the objective function. Two of them (0.0898, 0.7127), (-0.0898, -0.7127) are identified as the global optimal solutions of the 2-dimensional Hump Camel-Back function.

Table 1 The obtained decomposition points (DPs) and the local optimal solutions in Step 4.

DPs	Local optimal solutions	
(1.1092, 0.7683)	(0.0898, 0.7127)	(1.7036, 0.7961)
(1.2961, -0.6051)	(-0.0898,-0.7127)	(1.6071, -0.5687)
(-1.1092,-0.7683)	(-0.0898,-0.7127)	(-1.7036,-0.7961)
(-1.2961, 0.6051)	(0.0898, 0.7127)	(-1.6071, 0.5687)

Table 2 The obtained local optimal solutions and the objective function values of the 2-dimensional Hump Camel-Back function. The shaded lines are the global optimal solutions.

Local optimal solutions	$c(x_1, x_2)$
(0.0898, 0.7127)	-1.0316
(1.7036, 0.7961)	-0.2155
(-0.0898, -0.7127)	-1.0316
(-1.6071, 0.5687)	2.1043
(1.6071, -0.5687)	2.1043
(-1.7036, -0.7961)	-0.2155

B. 2-DIMENSIONAL GRIEWANK FUNCTION

Minimize the 2-dimensional Griewank function [18] described by

$$c(x_1, x_2) = a + b_1 x_1^2 + b_2 x_2^2 - d\cos(c_1 x_1)\cos(c_2 x_2) \quad \text{(9)}$$

where, a=1, $b_{1,2}$=1/200, c_1=1, c_2=1/$\sqrt{2}$, d=1.

The 2-dimensinal Griewank function has multiple optimal solutions and the origin is the global optimal solution.

Step 1: Construct the quasi-vector field

The quasi-vector field and the Jacobian matrix of the function (6) are follows.

Quasi-vector field:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2b_1 x_1 - dc_1 \sin(c_1 x_1)\cos(c_2 x_2) \\ -2b_2 x_2 - dc_2 \cos(c_1 x_1)\sin(c_2 x_2) \end{pmatrix} \quad \text{(10)}$$

Jacobian matrix J:

$$J = \begin{pmatrix} -2b_1 - dc_1^2 \cos(c_1 x_1)\cos(c_2 x_2) & dc_1 c_2 \sin(c_1 x_1)\sin(c_2 x_2) \\ dc_2 c_1 \sin(c_1 x_1)\sin(c_2 x_2) & -2b_2 - dc_2^2 \cos(c_1 x_1)\cos(c_2 x_2) \end{pmatrix} \quad \text{(11)}$$

Step 2: Obtain the first local optimal solution

Integrate the constructed quasi-vector field (10) from the initial condition (10.0, 10.0), and obtain the first local optimal solution (12.4407, 8.7097).

Step 3: Find decomposition points

Trace equilibrium points from the initial condition (12.4407, 8.7097), which is obtained as the local optimal solution in the Step 2, using Continuation method. Figure 6 shows an example of the bifurcation diagram when the parameter is b_1. The example shows that 4 equilibrium points are found from the intersection between the branches (paths) and b_1=1/200. The stability of the found equilibrium points is estimated by the eigenvalues of the Jacobian matrix (11). Follows are results.

Decomposition points: (14.0447, 6.4642), (11.0928, 6.8223), (10.8391, 10.9517).

We can obtain other equilibrium points on the quasi-stability boundary with the other parameter variation using the continuation method.

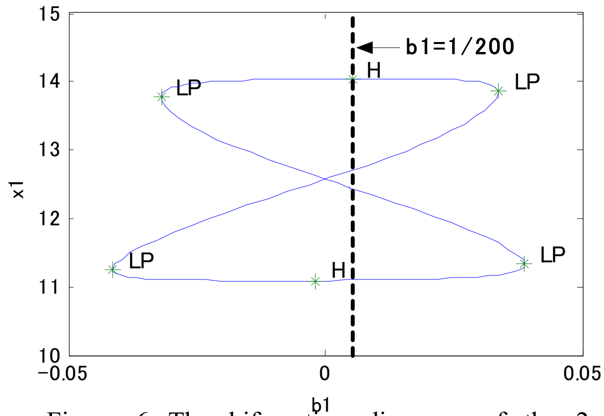


Figure 6 The bifurcation diagrams of the 2-dimenisional Griewank function when the parameter is b_1.

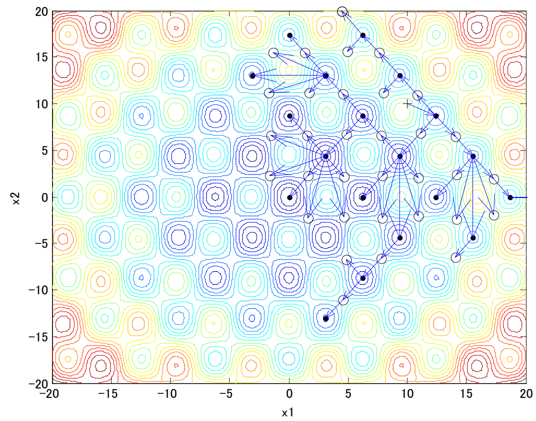


Figure 7 The obtained local optimal solutions (○), decomposition points (●), and the search paths (arrows) on the contour of the 2-dimensional Griewank function.

Step 4: Obtain the second, third, ...optimal solutions

Integrate the vector field from the intersection between the normalized unstable eigenvector of the decomposition points and the ε -ball. Follows are the obtained local optimal solutions.

Local optimal solutions: (15.5515, 4.3547), (9.3312, 4.3553), (9.3297, 13.0646).

Step 5: Identify the global optimal solution

We obtained the multiple local optimal solutions by above steps. Figure 7 shows the obtained local optimal solutions, decomposition points, and the search paths on the contour.

The origin is identified as the global optimal solutions of the 2-dimensional Griewank function.

VII. CONCLUSIONS

We presented a method to identify the locations of the decomposition points in the vector field and gave the novel approach to the searching of the global optimal solution for the general continuous nonlinear optimization problems. The proposed method overcomes the time-consuming issue of the previous method and has been verified by the 2-dimensional Hump Camel-Back function and the 2-dimensional Griewank function. The numerical results show the effectiveness of the proposed method. Expansion to the large-scale discrete and the mixed-integer nonlinear optimization problems is the future work.

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