Reaction of Trader Agent with Reinforcement Learning Mechanism in Artificial Market by Multi-agents System

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***Abstract*— The problem of optimal asset allocation is important an investor who desires to maxmize the expected utility concerning his consumption. Most conventional studies assumed the non-arbitrage condition in the perfect and efficient market and the period was only one. In this paper, we propose the multi-period consumption investment model by multi-agent system where agent's actions are reflected in a price of risk asset. And the agent learning is employed a reinforcement learning which an agent can get the environment and decide next action. By applying rational action of an investor to the agent, we build a virtually market on a computer. We statistically analyze the generated time series, compare to actual one. Moreover, we discuss the result of the participation of not only agent but also human being in artificial market.**

I. INTRODUCTION

Finance engineering is the field which developed quickly in recent years. It is supporting the deregulation and internationalization in a financial market from a theoretical viewpoint. Furthermore, with the advance of information technology, the finance theory has been applicable to the investment strategy which many financial products and assets management are incorporated into. As a result of this movement, the situation which surround a financial market has also been changing a lot [1]. A portfolio selection problem is one of the main subjects of finance engineering. This problem is that each investor determines that redistribution of his property will maximize the expected utility based on his consumption in the market which has non-risk assets and risk assets.

Many mathmatical models about the portfolio selection problem in the conventional finance theory are aimed at one period. However, it is desirable to assume that investment period is multiple. The reason is that actual investment needs periodical rebalance by change of a financial price, since it produces inflow and outflow of funds. Generally, it is difficult to search for the strict optimal portfolio strategy in a multi-period portfolio selection. Therefore, the portfolio strategy is formulated as the approximation model which assumed the non-arbitrage condition in the perfect and efficient market [2].

In the actual market, it is also difficult to confirm whether these assumptions are always satisfied. Moreover, the investor can not always reflect to all information quickly and appropriately. Then, many researches focus on the artificial market by the multi-agent system which a financial price is formed as a result of all investor's actions [4].

Because changing environment and interaction between investor's actions, it is difficult for an investor to be able to grasp the environment in detail and determine the optimal action. Therefore, it is needed the techniques which acquire the conduct code after much trial and error, and one of such techniques is reinforcement learning. In this research, we deal with the multi-period consumption investment model by the multi-agent, and propose the construction of the artificial market which an agent derives its portfolio strategy using reinforcement learning [6]. In the artificial model, we propose the decision making support system which analyze the influence of the expected utility depending on the financial price and consumption.

II. Outline of the multi-period consumption investment model

In this section, we explain a conventional multi-period consumption investment model (see Fig. 1) [3]. It is assumed that the market is perfect, that is, prices and profits are not affected. The amount of investment can take the arbitrary real numbers and short selling is allowed. There are cash and risk assets of J kinds. Concretely, asset 0 means a non-risk asset or call loan, and one from 1 to J means risk assets. Moreover, the investment begins at time 0 and finishes at time T. One period from time $(t-1)$ to t is called a period t. The investor decides redistribution of his property, satisfying the following conditions.

The sum of the investment ratio w_{j0} of risk asset j, the ratio c_0 of cash (call loan), and the ratio Φ_0 of consumption satisfies the following formulas at time 0.

$$\sum_{j=1}^{J} w_{j0} + c_0 + \Phi_0 = 1 \qquad (1)$$

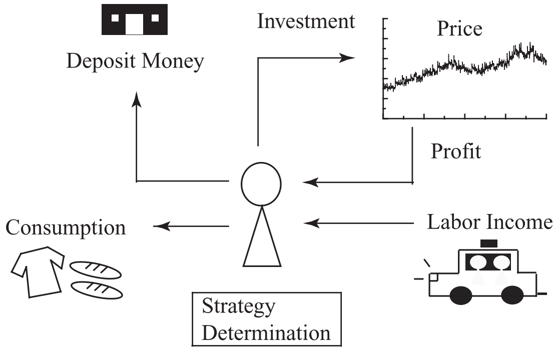

Fig. 1. Multi-period consumption investment model

By multiplying both sides by the initial property W_0 and using investment x_{j0} ($\equiv w_{j0}W_0$) of risk asset j, (1) becomes to

$$\sum_{j=1}^{J} x_{j0} + v_0 + \phi_0 = W_0$$

where v_0 ($\equiv c_0 W_0$) and ϕ_0 ($\equiv \Phi_0 W_0$) is initial cash and initial consumption, respectively. Moreover, using the price ρ_{j0} of the risk asset j and the amount of investment z_{j0} ($\equiv \frac{x_{j0}}{\rho_{j0}}$) of risk asset j at time 0, we obtain

$$\sum_{j=1}^{J} \rho_{j0} z_{j0} + v_0 + \phi_0 = W_0. \tag{2}$$

When a investor buys stocks z_{j0} of risk asset j at time 0, the value of risk asset j at time 1 is expressed as $\rho_{j1}z_{j0}$. The investor's wealth at time 1 is the sum of all risk assets, employment $(1+r_0)v_0$ of cash v_0 and labor income b_0. As a result of the distribution at time 0, the wealth at time 1 is given by

$$W_1 = \sum_{j=1}^{J} \rho_{j1} z_{j0} + (1+r_0)\, v_0 + b_0.$$

where r_0 is the interest rate of period 1. Similarly, based on the distribution at time $t-1$, the wealth W_t at time t is expressed as

$$\begin{aligned} W_t = & \sum_{j=1}^{J} (1+\mu_{jt})\rho_{j,t-1} z_{j,t-1} \\ & + (1+r_{t-1}) v_{t-1} + b_{t-1} \\ & t = 1, \ldots, T \end{aligned} \tag{3}$$

where μ_{jt} is the rate of return on the investment of risk asset j in period t, and r_{t-1} is the interest rate in period t. The distribution which the investor determined at time $t-1$ satisfies the following equation.

$$\begin{aligned} \sum_{j=1}^{J} \rho_{j,t-1} z_{j,t-1} + v_{t-1} = W_{t-1} - \phi_{t-1} \\ t = 1, \ldots, T \end{aligned} \tag{4}$$

Here, by the dividends on common stock and dividend of profit, the rate of return μ_{jt} of risk asset j is expressed as

$$\begin{aligned} \mu_{jt} &= \frac{\rho_{jt} - \rho_{j,t-1} + d_{jt}}{\rho_{j,t-1}} \\ &= \frac{\rho_{jt} - \rho_{j,t-1}}{\rho_{j,t-1}} + \frac{d_{jt}}{\rho_{j,t-1}}. \end{aligned} \tag{5}$$

where d_{jt} is the dividend or interest of risk asset j in period t. (3) is rewritten by (4) and (5)

$$\begin{aligned} W_t &= \sum_{j=1}^{J} \rho_{jt} z_{j,t-1} + \sum_{j=1}^{J} d_{jt} z_{j,t-1} \\ & \quad + (1+r_{t-1})\, v_{t-1} + b_{t-1}. \\ & \qquad t = 1, \ldots, T-1 \end{aligned} \tag{6}$$

Generally, the investor has to pay the brokerage commission because of the difference between purchase price and sale price. They bring together the transaction cost and securities transaction tax produced for the fluidity of market, and it is called the rate of the dealing cost. By the amount y^+_{jt} of purchase and the amount y^-_{jt} of sale, the amount of investment in risk asset j at time t is expressed as

$$\begin{aligned} z_{jt} &= z_{j,t-1} + y^+_{jt} - y^-_{jt}. \\ & \qquad j = 1, \ldots, J;\; t = 1, \ldots, T-1 \end{aligned} \tag{7}$$

If the rate of dealing cost takes a fixed value at all assets and all times, the purchase price will be the value which added dealing cost to the market price, and a sale price will be the value which deducted dealing cost from the market price. That is, when the rate of dealing cost is set to γ, the purchase price of risk asset j at time 0 becomes $(1+\gamma)\rho_{j0}$. Therefore, (2) which is the wealth at time 0 is rewritten by

$$\sum_{j=1}^{J} (1+\gamma)\rho_{j0} z_{j0} + v_0 + \phi_0 = W_0.$$

Similarly, (4) is rewritten by

$$\begin{aligned} & \sum_{j=1}^{J} \rho_{j,t-1} \left\{ z_{j,t-2} + (1+\gamma)\, y^+_{j,t-1} - (1-\gamma)\, y^-_{j,t-1} \right\} \\ & \qquad + v_{t-1} = W_{t-1} - \phi_{t-1} \\ & \sum_{j=1}^{J} \rho_{j,t-1} \left\{ z_{j,t-1} + \gamma \left(y^+_{j,t-1} + y^-_{j,t-1} \right) \right\} + v_{t-1} \\ & \qquad = W_{t-1} - \phi_{t-1} \\ & \qquad j = 1, \ldots, J;\; t = 1, \ldots, T-1 \end{aligned} \tag{8}$$

Each investor acts in order to maximize the following utility.

$$\begin{aligned} U(\phi_1, \ldots, \phi_\infty) &= u_1(\phi_1) + \alpha_1 u_2(\phi_2) + \ldots \\ & \quad + \alpha_1 \ldots \alpha_\infty u_\infty(\phi_\infty) \end{aligned}$$

where α is the discount rate for the future. Thus, the investor's purpose can be rewritten by

$$\text{Max E}[U(\phi_1, \ldots, \phi_T)]$$

for which the expected utility from a consumption series is maximized under $\phi_t \geq 0$. The utility function U is a monotonous increasing and concave. Also, it reflects the investor's preference

$$(a, b, \phi_3, \ldots, \phi_T) \geq (b, a, \phi_3, \ldots, \phi_T), \qquad (a > b)$$

where $U'_t > 0$, $U''_t > 0$ and $\alpha < 1$ about all t. One of utility functions is

$$u_t(\phi_t) = \frac{1}{\kappa}\phi_t^{\kappa} \qquad (\kappa < 1)$$

where κ represents the degree of his preference on the utility function.

III. The proposed of multiple period consumption investment model in artificial market

A. The outline of an artificial market by multi-agent system

Here, we relax some conditions of a usual models and describe the framework of the artificial market (see Fig. 2). It assumes that it is a discrete-time modeling, and all agents are noncooperative and rational. Non-risk asset ($j = 0$) and risk assets ($1 \leq j \leq J$) exist in a market. Furthermore, there are N agents modeling investors. Each agent need not behave uniformly.

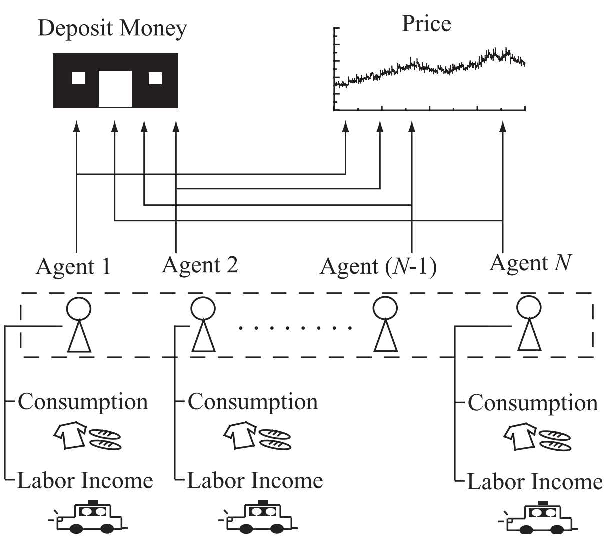

Fig. 2. Multi-agent system

The amount of investment in risk asset j of agent n in period t is described by $z_{jt}^{(n)}$, and agent n's consumption is $\phi_t^{(n)}$. In this artificial market, the company pays dividend d_{jt} to the agent holding its stocks. Therefore, the agent n's wealth at time t is written by

$$\begin{aligned} W_t &= \sum_{j=1}^{J} \rho_{jt} z_{j,t-1} + \sum_{j=1}^{J} r_d d_{jt} z_{j,t-1}^{(n)} \\ &\quad + (1 + r_{t-1})\, v_{t-1}^{(n)} + b_{t-1}^{(n)} \\ &\quad + (1 + \gamma) \sum_{j=1}^{J} z_{j,t-1}^{'(n)} \rho_{j,t-1} + \sum_{j=1}^{J} s_{jt}^{(n)}. \end{aligned}$$

where r_d is a positive constant, z'^{n}_{jt} is the repayment in the case of dealings failure and $b_{t-1}^{(n)}$ is the labor income which agent n obtains in period $t-1$. $s_{jt}^{(n)}$ is the amount of an excess or a deficiency accompanying price fluctuation.

$$\begin{aligned} s_{jt}^{(n)} = &\left(\rho_{jt} - \rho_{j,t-1}\right)(z_{j,t-1}^{(n)} - z_{j,t-1}^{'(n)}) \\ &\times sign(y_{jt}^{(n)-} - y_{jt}^{(n)+}) \end{aligned}$$

where $sign(x)$ is given by

$$sign(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

And the rate of return μ_{jt} is the same as (5). A dividend of risk asset j is the colored noise of at discrete time and it given by

$$\log \frac{d_{jt}}{\bar{d}_{jt}} = \epsilon_j^a \log \frac{d_{j,t-1}}{\bar{d}_{jt}} + \epsilon_j^b \xi_{jt}$$

where $\xi_{jt}(t)$ is the gaussian noise with an average 0 and a variance σ_j^2 [4]. And they are positive parameters which satisfy $(\epsilon_j^a)^2 + (\epsilon_j^b)^2 = 1$. $\log \frac{d_{jt}}{\bar{d}_{jt}}$ has an average 0 and a variance σ_j^2, its autocorrelation function decreases within the correlation time $\tau_s = 1/\log(\epsilon_j^a)$.

After the period is renewed, each agent changes its stocks and begins consuming. Here, the agent redistributes under the following condition that its property is fixed.

$$\begin{aligned} \sum_{j=1}^{J} \rho_{jt} \left\{ z_{jt}^{(n)} + \gamma \left(y_{jt}^{(n)+} + y_{jt}^{(n)-} \right) \right\} + v_t^{(n)} \\ = W_t^{(n)} - \phi_t^{(n)} \end{aligned}$$

And, it is assumued that the total amount of the risk asset j $(= 1, \ldots, J)$ in the artificial market is fixed.

$$\sum_{n=1}^{N} z_{jt}^{(n)} = Z_j$$

Each agent desires to maximize the following utility.

$$\begin{aligned} U(\phi_1^n, \ldots, \phi_\infty^{(n)}) &= u_1(\phi_1^{(n)}) + \alpha_1 u_2(\phi_2^{(n)}) + \ldots \\ &\quad + \alpha_1 \ldots \alpha_\infty u_\infty(\phi_\infty^{(n)}) \end{aligned}$$

By the way, each agent cannot always invest its stocks as requested. Here, agent n wants to buy $y_{jt}^{(n)+}$ or sell $y_{jt}^{(n)-}$ risk asset j, then the total amount of buying and selling at time t are derived by

$$\begin{aligned} B_{jt} &= \sum_{n=1}^{N} y_{jt}^{(n)+}, \\ O_{jt} &= \sum_{n=1}^{N} y_{jt}^{(n)-}. \end{aligned}$$

Therefore, in an only case of $B_{jt} = O_{jt}$, all agents can buy and sell stocks at the desire of them. On the other hand, in case of $B_{jt} \neq O_{jt}$, the amount of risk asset j which is actually moved is given by

$$z_{jt}^{(n)} = z_{j,t-1}^{(n)} + \frac{V_{jt}}{B_{jt}} y_{jt}^{(n)+} - \frac{V_{jt}}{O_{jt}} y_{jt}^{(n)-}$$

where $V_{jt} \equiv \min(B_{jt}, O_{jt})$. That is, the repayment $z'^{(n)}_{jt}$ in case of dealings failure is set to

$$z'^{(n)}_{jt} = \left\{ (1 - \frac{V_{jt}}{B_{jt}}) y_{jt}^{(n)+} - (1 - \frac{V_{jt}}{O_{jt}}) y_{jt}^{(n)-} \right\}$$

The price ρ_{jt} of risk asset j is determined based on the dealings of all agents. Under such a situation, price $\rho_{j,t+1}$ is given as follows.

$$\begin{aligned} \rho_{j,t+1} &= \frac{2\rho_{jt}}{1 + \exp\{-U_{jt}/T_j\}} \\ U_{jt} &= \log \frac{B_{jt}}{O_{jt}} \end{aligned}$$

where T_j is a positive constant and represents the sensitivity of risk asset j. When T_j is small value, agents are sensitive to the difference between demand and supply. Otherwise they are not sensitive to these values.

B. The portfolio selection by reinforcement learning

In an artificial market, agent n makes a decision based on the following reward

$$V_t^n = \sum_{k=0}^{\infty} (\alpha_n)^k \phi_{t+k}(\phi_{t+k}^n)$$

under finite property at time t. In this study, we propose the searching algorithm for the portfolio strategy by a reinforcement learning using neural networks. We apply the neural network [7] that realizes an actor-critic model [6] to a reinforcement learning.

First, agent n observes a state $\boldsymbol{x}_t^n$ about the environment. And, the actor model generates a control output by

$$\begin{aligned} q_{jt}^n &= f\left(\sum_{i=1}^{N_A} w_{ijt}^{A_n} g_i^{A_n}(\boldsymbol{x}_t^n) + n_{jt} \right) \\ g_i^{A_n}(\boldsymbol{x}_t^n) &= \exp\left\{ -\frac{1}{2} (\boldsymbol{x}_t^n - \boldsymbol{m}_i^{A_n})^{\mathrm{T}} \boldsymbol{C}_i^{A_n^{-1}} \right. \\ &\qquad \left. \times (\boldsymbol{x}_t^n - \boldsymbol{m}_i^{A_n}) \right\} \end{aligned}$$

where g_i^A is the i-th radial basis function, N_A is the number of radial basis functions, $w_{ijt}^{A_n}$ is the weight, and n_{jt} is standardized gaussian noise. f is sigmoid function

$$f(x) = \frac{g_i^{max}}{1 + \exp\{-x/T_n^A\}}$$

g_i^{max} is the maximum about output i, and T_n^A is the sensitivity of agent n.

In the actor model, agent desire to trade risk asset j by

$$\begin{aligned} y_{jt}^{(n)+} &= \frac{q'^n_{jt}\ W_t^{(n)}}{(1+\gamma)\rho_{jt}} - z_{j,t-1}^{(n)} \\ &\qquad (q'^n_{jt}\ W_t^{(n)} \geq (1+\gamma)\rho_{jt} z_{i,t-1}^{(n)}) \\ y_{jt}^{(n)-} &= z_{i,t-1}^{(n)} - \frac{q'^n_{jt}\ W_t^{(n)}}{(1+\gamma)\rho_{jt}} \\ &\qquad (q'^n_{jt}\ W_t^{(n)} < (1+\gamma)\rho_{jt} z_{i,t-1}^{(n)}) \end{aligned}$$

from a control output.

Next, the critic model generates the evaluation value

$$V_\pi^n(\boldsymbol{x}_t^n) = \sum_{i=1}^{N_C} w_{it}^{C_n} g_i^{C_n}(\boldsymbol{x}_t^n)$$

where N_C is the number of radial basis functions of the critic model. As a result of the agent's action, the critic model receives reward

$$R_t^n = u_t(\phi_t^{(n)})$$

from environment, and observes a state $\boldsymbol{x}_{t+1}^n$ after transition. The expected utility is given by

$$\mathrm{E}[u_t(\phi_t^n)] = V_\pi^n(\boldsymbol{x}_t^n) - \alpha_n V_\pi^n(\boldsymbol{x}_{t+1}^n)$$

Then the TD error which means a reinforcement signal is defined by the difference between the actual utility and the expected utility as follows.

$$\begin{aligned} \delta_t &\equiv u_t(\phi_t^n) - \mathrm{E}[u_t(c_t^n)] \\ &= u_t(\phi_t^n) + \alpha_n V_\pi^n(\boldsymbol{x}_{t+1}^n) - V_\pi^n(\boldsymbol{x}_t^n) \end{aligned}$$

After it is sent to the actor model, the past record of activity

$$e_{it}^n = \lambda_e e_{i,t-1}^n + g_i^{C_n}(\boldsymbol{x}_t^n)$$

is calculated. And, the weight is updated by

$$w_{it}^{C_n} = w_{i,t-1}^{C_n} + \eta_C \delta_t e_{it}^n$$

In the actor model, the weight is also updated by

$$w_{ijt}^{A_n} = w_{ij,t-1}^{A_n} + \eta_A \delta_t g_i^{A_n}(\boldsymbol{x}_t^n) n_{jt}$$

where η_A, η_C are the learning rates and λ_e is reduction.

IV. Simulation results and consideration

In this section, we show results that the reinforcement learning using neural networks apply to the portfolio strategy. Note that it is difficult to apply this artificial market to usual dynamic programming method. The values of simulation results are the average of 5 trials. It is assumed that one trial is 1224 steps, and last 1024 steps are observed.

First, we treat the artificial market which there are 20 agents, a non risk asset and risk assets of 3 kinds in. The sensitivity of each price is set to $T_1 = 900, T_2 = 750, T_3 = 600$. And the sensitivity of the agent is set to all $T_n = 240$ $(n = 1, 2, \ldots, 20)$. The initial values of risk assets which the agent has at time 0 are given by $1000000 + 50\sigma$, and the initial prices of risk assets are given by $1000 + 5\sigma$. Where, σ

TABLE I

THE VALUE OF PARAMETERS USED IN THIS SIMULATION

Parameter	Simulation's Value
η_A	0.01
η_C	0.01
λ_e	0.7
r_t	0.01
α	0.3
κ	0.5
γ	0.0001

TABLE II

RESULT OF THE STATISTICS

statistics	simulation result
Mean	−0.0001
Standard Deviation	0.00039
Skewness	−0.34
Kurtosis	1.76
First Order Autocorrelation	−0.524
Kolmogorov-Smirnov statistic	1.67
Jarque-Bera statistic	85.33
Box-Ljung statistic	280.953

is the uniform random number within $-1 \sim 1$. Moreover, it is shown in Table I that the values of the other parameters.

Fig. 3 and Fig. 4 show the dynamics of the prices of risk assets and the dynamics of the values of utility function for several agents in certain trial, respectively. In Fig. 4, the black line shows the value of agent whose number is 1, the right grey shows agent 10, and the dark grey shows agent 20.

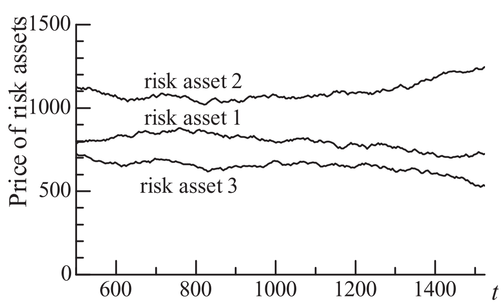

Fig. 3. One of the sample dynamics of the prices of risk assets

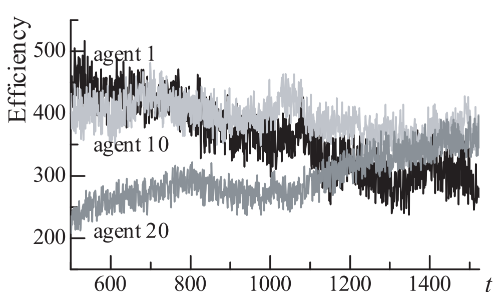

Fig. 4. One of the sample dynamics of the value of utility function for a certain agent

A. The comparison with the actual market

It is known that in actual stock markets the logarithm of rate of return does not follow the normal distribution. In order to indicate whether the proposed artificial market has the same properties as actual stock markets, we investigate the logarithm of rate of return. And, the result is shown in Table II

From Table II, it is observed that the logarithm of rate of return generated by proposed artificial market has rejected the hypothesis of following a normal distribution by the significance level of 5% . Moreover, by the Box-Ljung test it implies that the first order autocorrelation exists in this model. Therefore, it is thought that the artificial market which we proposed has the properties similar to the actual stock markets [8].

B. The Effect of the parameters on the agent's trading volume

Next, we investigate what impact the number of agents in the artificial market has on agent's trading volume. We change the number of agents from 10 to 50, and simulate in each case. The averages $\diamond$ and standard deviations of the agent's trading volume are shown in Fig. 5. Other parameters except the number of agents are the same values as previous simulation. Moreover, the value of each average is calculated per capita the trading volume at 1 step.

In Fig. 5, it is seen that the standard deviation decreases with increasing the number of agents, and trading volume hardly changes. When the number of agents is small, the system dynamics are strongly influenced by an agent's action than the case that many agents exist in the system. Therefore, we think that the trading volume becomes stable according to the increase in the number of agents.

Finally, we examine what effect the change of the rate of dealing cost γ has on the trading volume. After the trading volumes are calculated per risk asset at 1 step, its averages

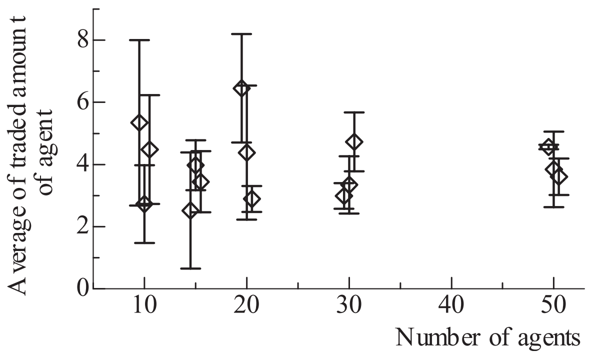

Fig. 5. The average of trading volume in which an agent trades at one step

and standard deviations about three trials are shown in Fig. 6. The number of agents is 20. And we use the same parameters as previous simulation.

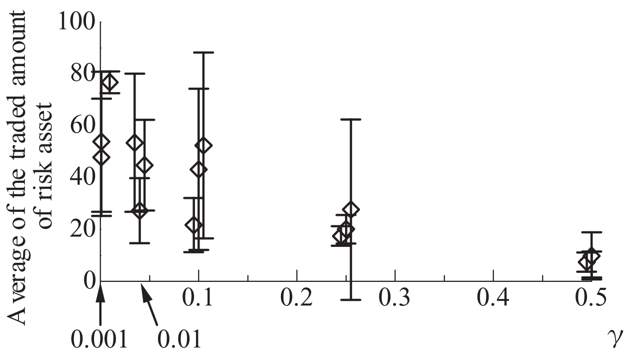

Fig. 6. The average of trading volume of a risk asset at one step

As shown this figure, the averages and standard deviations decrease as the rate of dealing cost increases. The reason is that it is hard for the agents to trade the risk assets, when the rate of dealing cost is high.

V. Conclusion

This study dealt with the situation where the interaction between investor's actions is complicated and their actions affect the environment. We proposed the multi-period consumption investment model by multi-agent system, in which each investor determines his action based on the information on the environment. Therefore, their actions are not uniform.

Then, we realized rational action of the investor on the computer virtually by applying the reinforcement learning to the agent whose action is modeled on the investor's one. Furthermore, we analyze that the number of agents and the rate of dealing cost influence to the trading volume. By refining the proposal technique, we hope that it can be useful for a development of a portfolio support system.

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