A New Similarity Measure Between Interval-Valued Fuzzy Numbers

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Abstract — **In this paper, we present a new method for calculating the degree of similarity between interval- valued trapezoidal fuzzy numbers. The proposed similarity measure uses the concept of geometry to calculate the center-of-gravity (COG) points of the lower trapezoidal fuzzy number and the upper trapezoidal fuzzy number of interval-valued trapezoidal fuzzy numbers, respectively, and then to calculate the degree of similarity between interval-valued trapezoidal fuzzy numbers. We also prove some properties of the proposed similarity measure. The proposed method provides a useful way to calculate the degree of similarity between interval-valued trapezoidal fuzzy numbers.**

I. Introduction

In [12], Guijun et al. presented the definition of interval-valued fuzzy numbers and their extended operations. In [16], Wang et al. studied the correlation coefficient of interval-valued fuzzy numbers and some of their properties. In [14], Lin used interval-valued fuzzy numbers to represent vague processing time for dealing with job-shop scheduling problems. In [17], Yao et al. used interval-valued fuzzy numbers to represent unknown job processing time for constructing a fuzzy flow-shop sequencing model. In [13], Hong et al. presented a distance measure of interval-valued fuzzy numbers. From [13], [14], [16] and [17], we can see that interval-valued fuzzy numbers are very useful to represent evaluating values to deal with real-world problems. From [5], we can see that the similarity measure between generalized fuzzy numbers [2] is very important in the research topics of fuzzy decision-making and fuzzy risk analysis.

In this paper, we present a new method to calculate the degree of similarity between interval-valued trapezoidal fuzzy numbers. Furthermore, we also prove some properties of the proposed similarity measure. The proposed method provides a useful way to calculate the degree of similarity between interval-valued trapezoidal fuzzy numbers.

The rest of this paper is organized as follows. In Section II, we briefly review the definitions of interval-valued fuzzy sets [11], interval-valued trapezoidal fuzzy numbers [13] and the simple center-of-gravity method (SCGM) [5]. In Section III, we present a new similarity measure of interval-valued trapezoidal fuzzy numbers. Furthermore, we also prove some properties of the proposed similarity measure. In Section IV, we use two examples for calculating the degrees of similarity between interval-valued trapezoidal fuzzy numbers. The conclusions are discussed in Section V.

II. Preliminaries

In this section, we briefly review the definitions of generalized trapezoidal fuzzy numbers [2], [3], interval-valued fuzzy sets [11], interval-valued fuzzy numbers [13] and the simple center-of-gravity method (SCGM) [5].

In [2] and [3], Chen represented a generalized trapezoidal fuzzy number $\tilde{A}$ as $\tilde{A} = (a_1, a_2, a_3, a_4; \hat{w}_{\tilde{A}})$, where $0 < \hat{w}_{\tilde{A}} \leq 1$ and a_1, a_2, a_3 and a_4 are real numbers. The generalized fuzzy number $\tilde{A}$ of the universe of discourse X is characterized by a membership function $\mu_{\tilde{A}}$, where $\mu_{\tilde{A}}: X \rightarrow [0, 1]$, as shown in Fig. 1.

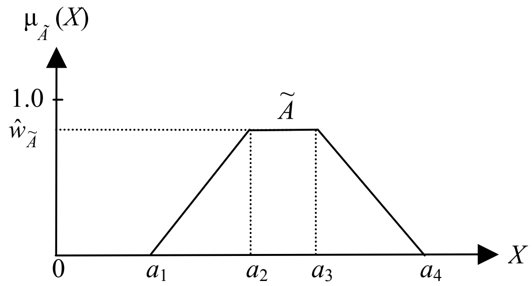

Fig. 1. A generalized trapezoidal fuzzy number $\tilde{A}$.

Definition 2.1: An interval-valued fuzzy set C defined in the universe of discourse X is defined by

$$C = \{(x, [\mu_C^L(x),\ \mu_C^U(x)]) \mid x \in X\},$$

where $0 \leq \mu_C^L(x) \leq \mu_C^U(x) \leq 1$ and the membership grade $\mu_C(x)$ of an element x belongs to the interval-valued fuzzy set C is represented by an interval $\mu_C(x) = [\mu_C^L(x), \mu_C^U(x)]$.

From [17], we can see that if $\tilde{\tilde{A}}^L = (a_1^L,\ a_2^L,\ a_3^L,\ a_4^L;\ \hat{w}_{\tilde{\tilde{A}}}^L)$ and $\tilde{\tilde{A}}^U = (a_1^U,\ a_2^U,\ a_3^U,\ a_4^U;\ \hat{w}_{\tilde{\tilde{A}}}^U)$, where $a_1^U \leq a_2^U \leq a_3^U \leq a_4^U$, $a_1^L \leq a_2^L \leq a_3^L \leq a_4^L$, $0 < \hat{w}_{\tilde{\tilde{A}}}^L \leq \hat{w}_{\tilde{\tilde{A}}}^U \leq 1$, and $\tilde{\tilde{A}}^L \subset \tilde{\tilde{A}}^U$, then the interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}} = [\tilde{\tilde{A}}^L,\ \tilde{\tilde{A}}^U] = [(a_1^L,\ a_2^L,\ a_3^L,\ a_4^L;\ \hat{w}_{\tilde{\tilde{A}}}^L), (a_1^U,\ a_2^U,\ a_3^U,\ a_4^U;\ \hat{w}_{\tilde{\tilde{A}}}^U)]$, as shown in Fig. 2.

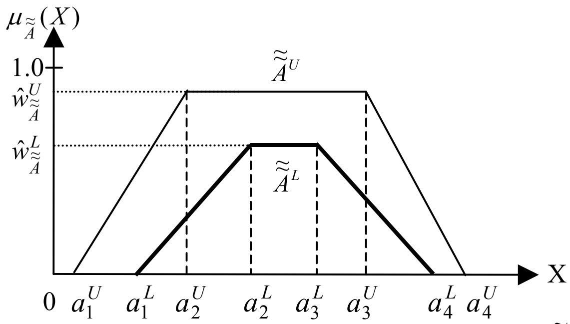

Fig. 2. Interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$.

From Fig. 2, we can see that the interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$ has two elements $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{A}}^U$, where $\tilde{\tilde{A}}^L$ is called the "lower trapezoidal fuzzy number", and $\tilde{\tilde{A}}^U$ is called the "upper trapezoidal fuzzy number". From Fig. 2, we can see that the two elements $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{A}}^U$ of the interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$ can be regarded as two different generalized fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{A}}^U$, where $\tilde{\tilde{A}}^L = (a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}^L_{\tilde{\tilde{A}}})$, $\tilde{\tilde{A}}^U = (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}^U_{\tilde{\tilde{A}}})$, $a_1^U \le a_2^U \le a_3^U \le a_4^U$, $a_1^L \le a_2^L \le a_3^L \le a_4^L$, $0 < \hat{w}^L_{\tilde{\tilde{A}}} \le \hat{w}^U_{\tilde{\tilde{A}}} \le 1$, and $\tilde{\tilde{A}}^L \subset \tilde{\tilde{A}}^U$. If $a_1^L = a_1^U$, $a_2^L = a_2^U$, $a_3^L = a_3^U$, $a_4^L = a_4^U$ and $\hat{w}^L_{\tilde{\tilde{A}}} = \hat{w}^U_{\tilde{\tilde{A}}} = \hat{w}_{\tilde{A}}$, then the interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$ can be regarded as a generalized trapezoidal fuzzy number, denoted as $\tilde{A} = (a_1, a_2, a_3, a_4; \hat{w}_{\tilde{A}})$.

Assume that there are two interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, where $\tilde{\tilde{A}} = [(a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}^L_{\tilde{\tilde{A}}}), (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}^U_{\tilde{\tilde{A}}})]$ and $\tilde{\tilde{B}} = [(b_1^L, b_2^L, b_3^L, b_4^L; \hat{w}^L_{\tilde{\tilde{B}}}), (b_1^U, b_2^U, b_3^U, b_4^U; \hat{w}^U_{\tilde{\tilde{B}}})]$. The arithmetic operations between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are defined as follows [8], [13]:

(1) Interval-Valued Trapezoidal Fuzzy Numbers Addition $\oplus$:

$$\tilde{\tilde{A}} \oplus \tilde{\tilde{B}} = [(a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}^L_{\tilde{\tilde{A}}}), (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}^U_{\tilde{\tilde{A}}})] \oplus [(b_1^L, b_2^L, b_3^L, b_4^L; \hat{w}^L_{\tilde{\tilde{B}}}), (b_1^U, b_2^U, b_3^U, b_4^U; \hat{w}^U_{\tilde{\tilde{B}}})]$$
$$= [(a_1^L + b_1^L, a_2^L + b_2^L, a_3^L + b_3^L, a_4^L + b_4^L; \text{Min}(\hat{w}^L_{\tilde{\tilde{A}}}, \hat{w}^L_{\tilde{\tilde{B}}})), (a_1^U + b_1^U, a_2^U + b_2^U, a_3^U + b_3^U, a_4^U + b_4^U; \text{Min}(\hat{w}^U_{\tilde{\tilde{A}}}, \hat{w}^U_{\tilde{\tilde{B}}}))], \tag{1}$$

where a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U, a_4^U, b_1^L, b_2^L, b_3^L, b_4^L, b_1^U, b_2^U, b_3^U and b_4^U are real values, $0 < \hat{w}^L_{\tilde{\tilde{A}}} \le \hat{w}^U_{\tilde{\tilde{A}}} \le 1$ and $0 < \hat{w}^L_{\tilde{\tilde{B}}} \le \hat{w}^U_{\tilde{\tilde{B}}} \le 1$.

(2) Interval-Valued Trapezoidal Fuzzy Numbers Multiplication $\otimes$:

$$\tilde{\tilde{A}} \otimes \tilde{\tilde{B}} = [(a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}^L_{\tilde{\tilde{A}}}), (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}^U_{\tilde{\tilde{A}}})] \otimes [(b_1^L, b_2^L, b_3^L, b_4^L; \hat{w}^L_{\tilde{\tilde{B}}}), (b_1^U, b_2^U, b_3^U, b_4^U; \hat{w}^U_{\tilde{\tilde{B}}})]$$
$$= [(a_1^L \times b_1^L, a_2^L \times b_2^L, a_3^L \times b_3^L, a_4^L \times b_4^L; \text{Min}(\hat{w}^L_{\tilde{\tilde{A}}}, \hat{w}^L_{\tilde{\tilde{B}}})), (a_1^U \times b_1^U, a_2^U \times b_2^U, a_3^U \times b_3^U, a_4^U \times b_4^U; \text{Min}(\hat{w}^U_{\tilde{\tilde{A}}}, \hat{w}^U_{\tilde{\tilde{B}}}))], \tag{2}$$

where a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U, a_4^U, b_1^L, b_2^L, b_3^L, b_4^L, b_1^U, b_2^U, b_3^U and b_4^U are real values, $0 < \hat{w}^L_{\tilde{\tilde{A}}} \le \hat{w}^U_{\tilde{\tilde{A}}} \le 1$ and $0 < \hat{w}^L_{\tilde{\tilde{B}}} \le \hat{w}^U_{\tilde{\tilde{B}}} \le 1$.

(3) Interval-Valued Trapezoidal Fuzzy Numbers Division $\varnothing$:

$$\tilde{\tilde{A}} \varnothing \tilde{\tilde{B}} = [(a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}^L_{\tilde{\tilde{A}}}), (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}^U_{\tilde{\tilde{A}}})] \varnothing [(b_1^L, b_2^L, b_3^L, b_4^L; \hat{w}^L_{\tilde{\tilde{B}}}), (b_1^U, b_2^U, b_3^U, b_4^U; \hat{w}^U_{\tilde{\tilde{B}}})]$$
$$= [(a_1^L / b_4^L, a_2^L / b_3^L, a_3^L / b_2^L, a_4^L / b_1^L; \text{Min}(\hat{w}^L_{\tilde{\tilde{A}}}, \hat{w}^L_{\tilde{\tilde{B}}})), (a_1^U / b_4^U, a_2^U / b_3^U, a_3^U / b_2^U, a_4^U / b_1^U; \text{Min}(\hat{w}^U_{\tilde{\tilde{A}}}, \hat{w}^U_{\tilde{\tilde{B}}}))], \tag{3}$$

where a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U, a_4^U, b_1^L, b_2^L, b_3^L, b_4^L, b_1^U, b_2^U, b_3^U and b_4^U are nonzero positive real values or nonzero negative real values, $0 < \hat{w}^L_{\tilde{\tilde{A}}} \le \hat{w}^U_{\tilde{\tilde{A}}} \le 1$ and $0 < \hat{w}^L_{\tilde{\tilde{B}}} \le \hat{w}^U_{\tilde{\tilde{B}}} \le 1$.

In [5], Chen et al. presented the simple center-of-gravity method (SCGM) to calculate the center-of-gravity point (x^*, y^*) of a generalized trapezoidal fuzzy number based on the concept of geometry. Let $\tilde{A}$ be a generalized trapezoidal fuzzy number, $\tilde{A} = (a_1, a_2, a_3, a_4; \hat{w}_{\tilde{A}})$. The formulas for calculating the COG point $(x^*_{\tilde{A}}, y^*_{\tilde{A}})$ of the generalized trapezoidal fuzzy number $\tilde{A}$ are as follows:

$$y^*_{\tilde{A}} = \begin{cases} \dfrac{\hat{w}_{\tilde{A}} \times (\frac{a_3 - a_2}{a_4 - a_1} + 2)}{6}, & \text{if } a_1 \ne a_4 \text{ and } 0 < \hat{w}_{\tilde{A}} \le 1, \\ \dfrac{\hat{w}_{\tilde{A}}}{2}, & \text{if } a_1 = a_4 \text{ and } 0 < \hat{w}_{\tilde{A}} \le 1, \end{cases} \tag{4}$$

$$x^*_{\tilde{A}} = \frac{y^*_{\tilde{A}}(a_3 + a_2) + (a_4 + a_1)(\hat{w}_{\tilde{A}} - y^*_{\tilde{A}})}{2\hat{w}_{\tilde{A}}}. \tag{5}$$

III. A New Method for Calculating the Degree of Similarity Between Interval-Valued Trapezoidal Fuzzy Numbers

In this section, we present a new similarity measure to calculate the degree of similarity between interval-valued trapezoidal fuzzy numbers. Let U be the universe of discourse, $U = [0, 1]$. Assume that there are two interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}} = [\tilde{\tilde{A}}^L, \tilde{\tilde{A}}^U] = [(a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}_{\tilde{\tilde{A}}}^L), (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}_{\tilde{\tilde{A}}}^U)]$ and $\tilde{\tilde{B}} = [\tilde{\tilde{B}}^L, \tilde{\tilde{B}}^U] = [(b_1^L, b_2^L, b_3^L, b_4^L; \hat{w}_{\tilde{\tilde{B}}}^L), (b_1^U, b_2^U, b_3^U, b_4^U; \hat{w}_{\tilde{\tilde{B}}}^U)]$, where $0 \le a_1^U \le a_2^U \le a_3^U \le a_4^U \le 1$, $0 \le a_1^L \le a_2^L \le a_3^L \le a_4^L \le 1$, $0 < \hat{w}_{\tilde{\tilde{A}}}^L \le \hat{w}_{\tilde{\tilde{A}}}^U \le 1$ and $\tilde{\tilde{A}}^L \subset \tilde{\tilde{A}}^U$; $0 \le b_1^U \le b_2^U \le b_3^U \le b_4^U \le 1$, $0 \le b_1^L \le b_2^L \le b_3^L \le b_4^L \le 1$, $0 < \hat{w}_{\tilde{\tilde{A}}}^L \le \hat{w}_{\tilde{\tilde{B}}}^U \le 1$ and $\tilde{\tilde{B}}^L \subset \tilde{\tilde{B}}^U$. The proposed method for calculating the degree of similarity between interval-valued trapezoidal fuzzy numbers is now presented as follows:

Step 1: Based on formulas (4) and (5), get two COG points of the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$, where the one is the COG point ($x^*_{\tilde{\tilde{A}}^L}$, $y^*_{\tilde{\tilde{A}}^L}$) of the lower trapezoidal fuzzy number $\tilde{\tilde{A}}^L$, and the other one is the COG point ($x^*_{\tilde{\tilde{A}}^U}$, $y^*_{\tilde{\tilde{A}}^U}$) of the upper trapezoidal fuzzy number $\tilde{\tilde{A}}^U$, shown as follows:

$$y^*_{\tilde{\tilde{A}}^L} = \begin{cases} \dfrac{\hat{w}_{\tilde{\tilde{A}}}^L \times \left(\dfrac{a_3^L - a_2^L}{a_4^L - a_1^L} + 2\right)}{6}, & \text{if } a_1^L \ne a_4^L \text{ and } 0 < \hat{w}_{\tilde{\tilde{A}}}^L \le 1, \\ \dfrac{\hat{w}_{\tilde{\tilde{A}}}^L}{2}, & \text{if } a_1^L = a_4^L \text{ and } 0 < \hat{w}_{\tilde{\tilde{A}}}^L \le 1, \end{cases} \tag{6}$$

$$x^*_{\tilde{\tilde{A}}^L} = \frac{y^*_{\tilde{\tilde{A}}^L}(a_3^L + a_2^L) + (a_4^L + a_1^L)(\hat{w}_{\tilde{\tilde{A}}}^L - y^*_{\tilde{\tilde{A}}^L})}{2\hat{w}_{\tilde{\tilde{A}}}^L}. \tag{7}$$

$$y^*_{\tilde{\tilde{A}}^U} = \begin{cases} \dfrac{\hat{w}_{\tilde{\tilde{A}}}^U \times \left(\dfrac{a_3^U - a_2^U}{a_4^U - a_1^U} + 2\right)}{6}, & \text{if } a_1^U \ne a_4^U \text{ and } 0 < \hat{w}_{\tilde{\tilde{A}}}^U \le 1, \\ \dfrac{\hat{w}_{\tilde{\tilde{A}}}^U}{2}, & \text{if } a_1^U = a_4^U \text{ and } 0 < \hat{w}_{\tilde{\tilde{A}}}^U \le 1, \end{cases} \tag{8}$$

$$x^*_{\tilde{\tilde{A}}^U} = \frac{y^*_{\tilde{\tilde{A}}^U}(a_3^U + a_2^U) + (a_4^U + a_1^U)(\hat{w}_{\tilde{\tilde{A}}}^U - y^*_{\tilde{\tilde{A}}^U})}{2\hat{w}_{\tilde{\tilde{A}}}^U}. \tag{9}$$

In the same way, we can get the COG points ($x^*_{\tilde{\tilde{B}}^L}$, $y^*_{\tilde{\tilde{B}}^L}$) and ($x^*_{\tilde{\tilde{B}}^U}$, $y^*_{\tilde{\tilde{B}}^U}$) of the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{B}}$.

Step 2: Calculate the degree of similarity $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L)$ between the lower trapezoidal fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{B}}^L$ and calculate the degree of similarity $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ between the upper trapezoidal fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$, shown as follows:

$$S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = \left[\left(1 - \frac{\sum_{i=1}^{4}\left|a_i^L - b_i^L\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\max(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}, \tag{10}$$

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = \left[\left(1 - \frac{\sum_{i=1}^{4}\left|a_i^U - b_i^U\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^U} - x^*_{\tilde{\tilde{B}}^U}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}{\max(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}, \tag{11}$$

where $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) \in [0, 1]$ and $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) \in [0, 1]$. The larger the value of $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L)$, the more the similarity between the lower trapezoidal fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{B}}^L$; the larger the value of $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$, the more the similarity between the upper trapezoidal fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$.

Step 3: Calculate the degree of similarity $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ as follows:

$$S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \sqrt{S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) \times S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)}, \tag{12}$$

where $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) \in [0, 1]$. The larger the value of $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$, the more the similarity between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$.

The proposed similarity measure between interval-valued trapezoidal fuzzy numbers has the following properties:

Property 3.1: Two interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are identical if and only if $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = 1$.

Proof:

(i) If $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are identical interval-valued trapezoidal fuzzy numbers, then $a_1^L = b_1^L$, $a_2^L = b_2^L$, $a_3^L = b_3^L$, $a_4^L = b_4^L$, $a_1^U = b_1^U$, $a_2^U = b_2^U$, $a_3^U = b_3^U$, $a_4^U = b_4^U$, $\hat{w}_{\tilde{\tilde{A}}}^L = \hat{w}_{\tilde{\tilde{B}}}^L$ and $\hat{w}_{\tilde{\tilde{A}}}^U = \hat{w}_{\tilde{\tilde{B}}}^U$. Based on formulas (10) and (11), the degree of similarity $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L)$ between the lower trapezoidal fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{B}}^L$ and the degree of similarity $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ between the trapezoidal upper fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$ can be calculated as follows:

$$\begin{aligned} S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) &= \left[\left(1 - \frac{\sum_{i=1}^{4}\left|a_i^L - b_i^L\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\max(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})} \\ &= [(1-0) \times (1-0)]^{\frac{1}{2}} \times 1 \\ &= 1, \end{aligned}$$

$$\begin{aligned} S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) &= \left[\left(1 - \frac{\sum_{i=1}^{4}\left|a_i^U - b_i^U\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^U} - x^*_{\tilde{\tilde{B}}^U}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}{\max(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})} \\ &= [(1-0) \times (1-0)]^{\frac{1}{2}} \times 1 \\ &= 1. \end{aligned}$$

Therefore, we can get

$$\begin{aligned} S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) &= \sqrt{S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) \times S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)} \\ &= \sqrt{1 \times 1} \\ &= 1. \end{aligned}$$

(ii) If $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = 1$, then $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = 1$ and $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = 1$. Based on formula (10), we can see that if $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = 1$, then

$$S(\widetilde{\widetilde{A}}^L, \widetilde{\widetilde{B}}^L) = [(1 - \frac{\sum_{i=1}^{4}\left|a_i^L - b_i^L\right|}{4}) \times (1 - \left|x^*_{\widetilde{\widetilde{A}}^L} - x^*_{\widetilde{\widetilde{B}}^L}\right|)]^{\frac{1}{2}} \times \frac{\min(y^*_{\widetilde{\widetilde{A}}^L}, y^*_{\widetilde{\widetilde{B}}^L})}{\max(y^*_{\widetilde{\widetilde{A}}^L}, y^*_{\widetilde{\widetilde{B}}^L})} = 1.$$

It implies that $a_1^L = b_1^L$, $a_2^L = b_2^L$, $a_3^L = b_3^L$, $a_4^L = b_4^L$, $x^*_{\widetilde{\widetilde{A}}^L} = x^*_{\widetilde{\widetilde{B}}^L}$, $y^*_{\widetilde{\widetilde{A}}^L} = y^*_{\widetilde{\widetilde{B}}^L}$ and $\hat{w}^L_{\widetilde{\widetilde{A}}} = \hat{w}^L_{\widetilde{\widetilde{B}}}$. Therefore, the lower trapezoidal fuzzy numbers $\widetilde{\widetilde{A}}^L$ and $\widetilde{\widetilde{B}}^L$ are identical. In the same way, if $S(\widetilde{\widetilde{A}}^U, \widetilde{\widetilde{B}}^U)$ = 1, then based on formula (11), we can see that

$$S(\widetilde{\widetilde{A}}^U, \widetilde{\widetilde{B}}^U) = [(1 - \frac{\sum_{i=1}^{4}\left|a_i^U - b_i^U\right|}{4}) \times (1 - \left|x^*_{\widetilde{\widetilde{A}}^U} - x^*_{\widetilde{\widetilde{B}}^U}\right|)]^{\frac{1}{2}} \times \frac{\min(y^*_{\widetilde{\widetilde{A}}^U}, y^*_{\widetilde{\widetilde{B}}^U})}{\max(y^*_{\widetilde{\widetilde{A}}^U}, y^*_{\widetilde{\widetilde{B}}^U})} = 1.$$

It implies that $a_1^U = b_1^U$, $a_2^U = b_2^U$, $a_3^U = b_3^U$, $a_4^U = b_4^U$, $x^*_{\widetilde{\widetilde{A}}^U} = x^*_{\widetilde{\widetilde{B}}^U}$, $y^*_{\widetilde{\widetilde{A}}^U} = y^*_{\widetilde{\widetilde{B}}^U}$ and $\hat{w}^U_{\widetilde{\widetilde{A}}} = \hat{w}^U_{\widetilde{\widetilde{B}}}$. Therefore, the upper trapezoidal fuzzy numbers $\widetilde{\widetilde{A}}^U$ and $\widetilde{\widetilde{B}}^U$ are identical.

From (i) and (ii), we can see that the interval-valued trapezoidal fuzzy numbers $\widetilde{\widetilde{A}}$ and $\widetilde{\widetilde{B}}$ are identical. Q.E.D.

Property 3.2: $S(\widetilde{\widetilde{A}}, \widetilde{\widetilde{B}}) = S(\widetilde{\widetilde{B}}, \widetilde{\widetilde{A}})$.

Proof: Based on formula (12), we can see that

$$S(\widetilde{\widetilde{A}}, \widetilde{\widetilde{B}}) = \sqrt{S(\widetilde{\widetilde{A}}^L, \widetilde{\widetilde{B}}^L) \times S(\widetilde{\widetilde{A}}^U, \widetilde{\widetilde{B}}^U)},$$

$$S(\widetilde{\widetilde{B}}, \widetilde{\widetilde{A}}) = \sqrt{S(\widetilde{\widetilde{B}}^L, \widetilde{\widetilde{A}}^L) \times S(\widetilde{\widetilde{B}}^U, \widetilde{\widetilde{A}}^U)},$$

where

$$S(\widetilde{\widetilde{A}}^L, \widetilde{\widetilde{B}}^L) = [(1 - \frac{\sum_{i=1}^{4}\left|a_i^L - b_i^L\right|}{4}) \times (1 - \left|x^*_{\widetilde{\widetilde{A}}^L} - x^*_{\widetilde{\widetilde{B}}^L}\right|)]^{\frac{1}{2}} \times \frac{\min(y^*_{\widetilde{\widetilde{A}}^L}, y^*_{\widetilde{\widetilde{B}}^L})}{\max(y^*_{\widetilde{\widetilde{A}}^L}, y^*_{\widetilde{\widetilde{B}}^L})},$$

$$S(\widetilde{\widetilde{B}}^L, \widetilde{\widetilde{A}}^L) = [(1 - \frac{\sum_{i=1}^{4}\left|b_i^L - a_i^L\right|}{4}) \times (1 - \left|x^*_{\widetilde{\widetilde{B}}^L} - x^*_{\widetilde{\widetilde{A}}^L}\right|)]^{\frac{1}{2}} \times \frac{\min(y^*_{\widetilde{\widetilde{B}}^L}, y^*_{\widetilde{\widetilde{A}}^L})}{\max(y^*_{\widetilde{\widetilde{B}}^L}, y^*_{\widetilde{\widetilde{A}}^L})},$$

$$S(\widetilde{\widetilde{A}}^U, \widetilde{\widetilde{B}}^U) = [(1 - \frac{\sum_{i=1}^{4}\left|a_i^U - b_i^U\right|}{4}) \times (1 - \left|x^*_{\widetilde{\widetilde{A}}^U} - x^*_{\widetilde{\widetilde{B}}^U}\right|)]^{\frac{1}{2}} \times \frac{\min(y^*_{\widetilde{\widetilde{A}}^U}, y^*_{\widetilde{\widetilde{B}}^U})}{\max(y^*_{\widetilde{\widetilde{A}}^U}, y^*_{\widetilde{\widetilde{B}}^U})},$$

$$S(\widetilde{\widetilde{B}}^U, \widetilde{\widetilde{A}}^U) = [(1 - \frac{\sum_{i=1}^{4}\left|b_i^U - a_i^U\right|}{4}) \times (1 - \left|x^*_{\widetilde{\widetilde{B}}^U} - x^*_{\widetilde{\widetilde{A}}^U}\right|)]^{\frac{1}{2}} \times \frac{\min(y^*_{\widetilde{\widetilde{B}}^U}, y^*_{\widetilde{\widetilde{A}}^U})}{\max(y^*_{\widetilde{\widetilde{B}}^U}, y^*_{\widetilde{\widetilde{A}}^U})},$$

where $\sum_{i=1}^{4}\left|a_i^L - b_i^L\right| = \sum_{i=1}^{4}\left|b_i^L - a_i^L\right|$, $\sum_{i=1}^{4}\left|a_i^U - b_i^U\right| = \sum_{i=1}^{4}\left|b_i^U - a_i^U\right|$, $\left|x^*_{\widetilde{\widetilde{A}}^L} - x^*_{\widetilde{\widetilde{B}}^L}\right| = \left|x^*_{\widetilde{\widetilde{B}}^L} - x^*_{\widetilde{\widetilde{A}}^L}\right|$, $\left|x^*_{\widetilde{\widetilde{A}}^U} - x^*_{\widetilde{\widetilde{B}}^U}\right| = \left|x^*_{\widetilde{\widetilde{B}}^U} - x^*_{\widetilde{\widetilde{A}}^U}\right|$, $\frac{\min(y^*_{\widetilde{\widetilde{A}}^L}, y^*_{\widetilde{\widetilde{B}}^L})}{\max(y^*_{\widetilde{\widetilde{A}}^L}, y^*_{\widetilde{\widetilde{B}}^L})} = \frac{\min(y^*_{\widetilde{\widetilde{B}}^L}, y^*_{\widetilde{\widetilde{A}}^L})}{\max(y^*_{\widetilde{\widetilde{B}}^L}, y^*_{\widetilde{\widetilde{A}}^L})}$ and $\frac{\min(y^*_{\widetilde{\widetilde{A}}^U}, y^*_{\widetilde{\widetilde{B}}^U})}{\max(y^*_{\widetilde{\widetilde{A}}^U}, y^*_{\widetilde{\widetilde{B}}^U})} = \frac{\min(y^*_{\widetilde{\widetilde{B}}^U}, y^*_{\widetilde{\widetilde{A}}^U})}{\max(y^*_{\widetilde{\widetilde{B}}^U}, y^*_{\widetilde{\widetilde{A}}^U})}$. Thus, we can see that $S(\widetilde{\widetilde{A}}^L, \widetilde{\widetilde{B}}^L) = S(\widetilde{\widetilde{B}}^L, \widetilde{\widetilde{A}}^L)$ and $S(\widetilde{\widetilde{A}}^U, \widetilde{\widetilde{B}}^U) = S(\widetilde{\widetilde{B}}^U, \widetilde{\widetilde{A}}^U)$. Therefore, $S(\widetilde{\widetilde{A}}, \widetilde{\widetilde{B}}) = S(\widetilde{\widetilde{B}}, \widetilde{\widetilde{A}})$. Q.E.D.

Property 3.3: If $\widetilde{\widetilde{A}}$ and $\widetilde{\widetilde{B}}$ be real values between zero and one, where $\widetilde{\widetilde{A}} = a$ and $\widetilde{\widetilde{B}} = b$, then $S(\widetilde{\widetilde{A}}, \widetilde{\widetilde{B}}) = 1 - |a - b|$.

Proof: If $\widetilde{\widetilde{A}}$ and $\widetilde{\widetilde{B}}$ be real values, then we can see that

$$\begin{aligned}\widetilde{\widetilde{A}} &= [(a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}^L_{\widetilde{\widetilde{A}}}), (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}^U_{\widetilde{\widetilde{A}}})] \\ &= [(a, a, a, a; 1), (a, a, a, a; 1)] \\ &= (a, a, a, a; 1) \\ &= a,\end{aligned}$$

$$\begin{aligned}\widetilde{\widetilde{B}} &= [(b_1^L, b_2^L, b_3^L, b_4^L; \hat{w}^L_{\widetilde{\widetilde{B}}}), (b_1^U, b_2^U, b_3^U, b_4^U; \hat{w}^U_{\widetilde{\widetilde{B}}})] \\ &= [(b, b, b, b; 1), (b, b, b, b; 1)] \\ &= (b, b, b, b; 1) \\ &= b.\end{aligned}$$

Based on formulas (6) and (8), we can see that $y^*_{\widetilde{\widetilde{A}}^L} = y^*_{\widetilde{\widetilde{B}}^L} = y^*_{\widetilde{\widetilde{A}}^U} = y^*_{\widetilde{\widetilde{B}}^U}$ = 1/2. Based on formulas (10) and (11), we can get

$$\begin{aligned}S(\widetilde{\widetilde{A}}^L, \widetilde{\widetilde{B}}^L) &= [(1 - \frac{\sum_{i=1}^{4}\left|a_i^L - b_i^L\right|}{4}) \times (1 - \left|x^*_{\widetilde{\widetilde{A}}^L} - x^*_{\widetilde{\widetilde{B}}^L}\right|)]^{\frac{1}{2}} \times \frac{\min(y^*_{\widetilde{\widetilde{A}}^L}, y^*_{\widetilde{\widetilde{B}}^L})}{\max(y^*_{\widetilde{\widetilde{A}}^L}, y^*_{\widetilde{\widetilde{B}}^L})} \\ &= [(1 - \frac{4|a - b|}{4}) \times (1 - |a - b|)]^{\frac{1}{2}} \times 1 \\ &= 1 - |a - b|,\end{aligned}$$

$$S(\widetilde{\widetilde{A}}^U, \widetilde{\widetilde{B}}^U) = [(1 - \frac{\sum_{i=1}^{4}\left|a_i^U - b_i^U\right|}{4}) \times (1 - \left|x^*_{\widetilde{\widetilde{A}}^U} - x^*_{\widetilde{\widetilde{B}}^U}\right|)]^{\frac{1}{2}} \times \frac{\min(y^*_{\widetilde{\widetilde{A}}^U}, y^*_{\widetilde{\widetilde{B}}^U})}{\max(y^*_{\widetilde{\widetilde{A}}^U}, y^*_{\widetilde{\widetilde{B}}^U})}$$

$$= [(1-\frac{4|a-b|}{4}) \times (1-|a-b|)]^{\frac{1}{2}} \times 1$$
$$= 1-|a-b|.$$

Therefore, we can see that

$$S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \sqrt{S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) \times S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)}$$
$$= \sqrt{(1-|a-b|) \times (1-|a-b|)}$$
$$= 1-|a-b|.$$

Q.E.D.

IV. NUMERICAL EXAMPLES

In the following, we use two examples to calculate the degrees of similarity between interval-valued trapezoidal fuzzy numbers.

Example 4.1: Assume that there are two interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, shown as follows:

$$\tilde{\tilde{A}} = [(0.3, 0.325, 0.375, 0.4; 0.25), \quad (0.1, 0.2, 0.3, 0.4; 1)],$$
$$\tilde{\tilde{B}} = [(0.5, 0.525, 0.575, 0.6; 0.25), \quad (0.5, 0.6, 0.7, 0.8; 1)].$$

The process for calculating the degree of similarity between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is illustrated as follows:

[Step 1] Based on formulas (6)-(9), we can get the COG point COG($\tilde{\tilde{A}}^L$) = ($x^*_{\tilde{\tilde{A}}^L}$, $y^*_{\tilde{\tilde{A}}^L}$) of the lower trapezoidal fuzzy numbers $\tilde{\tilde{A}}^L$, where

$$y^*_{\tilde{\tilde{A}}^L} = \frac{\hat{w}^L_{\tilde{\tilde{A}}} \times (\frac{a^L_3 - a^L_2}{a^L_4 - a^L_1} + 2)}{6}$$
$$= \frac{0.25 \times (\frac{0.375-0.325}{0.4-0.3} + 2)}{6}$$
$$= 0.1042,$$

$$x^*_{\tilde{\tilde{A}}^L} = \frac{y^*_{\tilde{\tilde{A}}^L}(a^L_3 + a^L_2) + (a^L_4 + a^L_1)(\hat{w}^L_{\tilde{\tilde{A}}} - y^*_{\tilde{\tilde{A}}^L})}{2\hat{w}^L_{\tilde{\tilde{A}}}}$$
$$= \frac{0.1042 \times (0.375 + 0.325) + (0.4 + 0.3) \times (0.25 - 0.1042)}{2 \times 0.25}$$
$$= 0.35.$$

In the same way, we can get the COG point COG($\tilde{\tilde{B}}^L$) = (0.55, 0.1042) of the lower trapezoidal fuzzy numbers $\tilde{\tilde{B}}^L$; we can get the COG point COG($\tilde{\tilde{A}}^U$) = (0.25, 0.3889) of the upper trapezoidal fuzzy numbers $\tilde{\tilde{A}}^U$; we can get the COG point COG($\tilde{\tilde{B}}^U$) = (0.65, 0.3889) of the upper trapezoidal fuzzy numbers $\tilde{\tilde{B}}^U$.

[Step 2] Based on formula (10), we can calculate the degree of similarity $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L)$ between the lower trapezoidal fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{B}}^L$, shown as follows:

$$S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = [(1 - \frac{\sum_{i=1}^{4}|a^L_i - b^L_i|}{4}) \times (1 - |x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L}|)]^{\frac{1}{2}}$$
$$\times \frac{\min(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\max(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}$$
$$= [(1 - \frac{|0.3-0.5| + |0.325-0.525| + |0.375-0.575| + |0.4-0.6|}{4})$$
$$\times (1 - |0.35 - 0.55|) \times 1]$$
$$= 0.8.$$

Based on formula (11), we can calculate the degree of similarity $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ between the upper trapezoidal fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$, where $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = 0.6$.

[Step 3] Based on formula (12), we can calculate the degree of similarity $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, shown as follows:

$$S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \sqrt{S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) \times S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)}$$
$$= \sqrt{0.8 \times 0.6}$$
$$\fallingdotseq 0.6928.$$

That is, the degree of similarity between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is about 0.6928.

Example 4.2: Assume that $\tilde{\tilde{A}}$ is an interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$, and assume that $\tilde{B}$ is a generalized trapezoidal fuzzy number, where

$$\tilde{\tilde{A}} = [(0.1, 0.2, 0.3, 0.4; 0.5), (0.1, 0.2, 0.3, 0.4; 1)],$$
$$\tilde{B} = (0.4, 0.5, 0.6, 0.7; 0.5).$$

Based on the discussions of Section 2, we can see that the generalized trapezoidal fuzzy number $\tilde{B}$ can be represented as follows:

$$\tilde{B} = (0.4, 0.5, 0.6, 0.7; 0.5)$$
$$= [(0.4, 0.5, 0.6, 0.7; 0.5), (0.4, 0.5, 0.6, 0.7; 0.5)].$$

The process for calculating the degree of similarity between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and the generalized trapezoidal fuzzy number $\tilde{B}$ is illustrated as follows:

[Step 1] Based on formulas (6)-(9), we can get the COG point COG($\tilde{\tilde{A}}^L$) = ($x^*_{\tilde{\tilde{A}}^L}$, $y^*_{\tilde{\tilde{A}}^L}$) of the lower trapezoidal fuzzy numbers $\tilde{\tilde{A}}^L$, shown as follows:

$$y^*_{\tilde{\tilde{A}}^L} = \frac{0.5 \times (\frac{0.3\text{-}0.2}{0.4\text{-}0.1} + 2)}{6}$$
$$= 0.1944,$$

$$x^*_{\tilde{\tilde{A}}^L} = \frac{0.1944 \times (0.3 + 0.2) + (0.4 + 0.1) \times (0.5 - 0.1944)}{2 \times 0.5}$$
$$= 0.25.$$

In the same way, we can get the COG points $\text{COG}(\widetilde{\widetilde{A}}^U)$ = (0.25, 0.3889) of the interval-valued trapezoidal fuzzy numbers $\widetilde{\widetilde{A}}$. Based on formulas (4) and (5), we can get the COG point $\text{COG}(\widetilde{B})$ of the generalized trapezoidal fuzzy number $\widetilde{B}$, where $\text{COG}(\widetilde{B}) = (0.55, 0.1944)$.

[Step 2] Based on formula (10), we can calculate the degree of similarity $S(\widetilde{\widetilde{A}}^L, \widetilde{B})$ between the interval-valued trapezoidal fuzzy numbers $\widetilde{\widetilde{A}}$ and the generalized trapezoidal fuzzy number $\widetilde{B}$, shown as follows:

$$\begin{aligned} S(\widetilde{\widetilde{A}}^L, \widetilde{B}) &= [(1 - \frac{|0.1-0.4| + |0.2-0.5| + |0.3-0.6| + |0.4-0.7|}{4}) \times \\ &\quad (1 - |0.25 - 0.55|) \times 1] \\ &= 0.7. \end{aligned}$$

In the same way, based on formula (11), we can calculate the degree of similarity $S(\widetilde{\widetilde{A}}^U, \widetilde{B})$ between the interval-valued trapezoidal fuzzy numbers $\widetilde{\widetilde{A}}$ and the generalized trapezoidal fuzzy number $\widetilde{B}$, where $S(\widetilde{\widetilde{A}}^U, \widetilde{B}) = 0.35$.

[Step 3] Based on formula (12), we can calculate the degree of similarity $S(\widetilde{\widetilde{A}}, \widetilde{B})$ between the interval-valued trapezoidal fuzzy numbers $\widetilde{\widetilde{A}}$ and the generalized trapezoidal fuzzy number $\widetilde{B}$, shown as follows:

$$\begin{aligned} S(\widetilde{\widetilde{A}}, \widetilde{B}) &= \sqrt{S(\widetilde{\widetilde{A}}^L, \widetilde{B}) \times S(\widetilde{\widetilde{A}}^U, \widetilde{B})} \\ &= \sqrt{0.7 \times 0.35} \\ &\fallingdotseq 0.495. \end{aligned}$$

That is, the degree of similarity between the interval-valued trapezoidal fuzzy numbers $\widetilde{\widetilde{A}}$ and the generalized trapezoidal fuzzy number $\widetilde{B}$ is about 0.495.

V. CONCLUSIONS

In this paper, we have presented a new method to calculate the degree of similarity between interval-valued trapezoidal fuzzy numbers. We also have proved some properties of the proposed similarity measure. The proposed similarity measure provides a useful way to calculate the degree of similarity between internal-valued trapezoidal fuzzy numbers.

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