Extending the Measure of Rough Dependency for Fuzzy Classification

Van Nam Huynh
School of Knowledge Science
JAIST Hokuriku
Ishikawa 923-1292, Japan
E-mail: huynh@jaist.ac.jp

Tetsuya Murai
Graduate School of Engineering
Hokkaido University
Sapporo 060-8628, Japan
E-mail: murahiko@main.eng.hokudai.ac.jp

Tu Bao Ho, Yoshiteru Nakamori
School of Knowledge Science
JAIST Hokuriku
Ishikawa 923-1292, Japan
E-mail: nakamori@jaist.ac.jp

***Abstract*— In rough-set-based data analysis, the so-called approximation quality is the traditional measure to evaluate the classification success of attributes in terms of a numerical evaluation of the dependency properties generated by these attributes. To deal with practical situations where a fuzzy classification must be approximated by available knowledge expressed in terms of a Pawlak's approximation space, we introduce in this paper an extension of this measure aimed at providing a numerical characteristic for such situations. Other related coefficients as *precision* and *significance* are also discussed correspondingly. A simple example is given to illustrate the proposed notions.**

I. Introduction

After nearly twenty years of introducing fuzzy sets [19], the notion of a rough set [12] has been introduced as a new mathematical tool to deal with the approximation of a concept in the context of incomplete information. Basically, while a fuzzy set introduced by Zadeh models the ill-definition of the boundary of a concept often described linguistically, a rough set introduced by Pawlak characterizes a concept by its lower and upper approximations due to indiscernibility between objects arose because of incompleteness of available knowledge. Both the theories have been proving to be of substantial importance in many areas of application [10], [11], [13], [15], [20].

Since the introduction of rough set theory, many attempts to establish the relationships between the two theories, to compare each to the other, and to simultaneously hybridize them have been made (e.g. [7], [11], [14], [16], [17], [18]). As an attempt in the line of integration between the two theories, Banerjee and Pal [3] have recently proposed a roughness measure for fuzzy sets, making use of the concept of a rough fuzzy set [7]. However, as pointed out in [9], Banerjee and Pal's roughness measure exhibits some undesired properties. Very recently, the authors in [9] have introduced an alternative roughness measure for fuzzy sets based on the notions of the mass assignment of a fuzzy set and its α-cuts. It has been shown that this new measure of roughness satisfies interesting properties and simultaneously avoids these undesired properties.

As is well-known, in rough-set-based data analysis, the so-called approximation quality measure is often used to evaluate the classification success of attributes in terms of a numerical evaluation of the dependency properties generated by these attributes. To deal with practical situations where a fuzzy classification must be approximated by available knowledge expressed in terms of a Pawlak's approximation space, we introduce in this paper an extension of approximation quality measure aimed at providing a numerical characteristic for such situations. Furthermore, extensions of related coefficients such as the precision measure and the significance measure are also discussed.

The rest of this paper is organized as follows. Section II briefly introduces necessary notions of rough sets and fuzzy sets, the mass assignment of a fuzzy set. In Section III, after recalling the notion of a rough fuzzy set [7] roughness measures of a fuzzy set are briefly reviewed. Section IV discusses an extension of the approximation quality measure to deal with situations where a fuzzy classification must be approximated by available knowledge expressed in terms of an approximation space. An illustration example is presented in Section V. Finally, some concluding remarks are presented in Section VI.

II. Preliminaries

In this section we recall basic notions in the theories of rough sets and fuzzy sets. Throughout this paper, we suppose that U is a finite non-empty set.

A. Pawlak's Approximation Quality

The rough set theory begins with the notion of an approximation space, which is a pair $\langle U, R\rangle$, where U be the universe of discourse and R an equivalence relation on U, i.e., R is reflexive, symmetric, and transitive. The relation R decomposes the set U into disjoint classes in such a way that two elements x, y are in the same class iff $(x, y) \in R$. Let denote by U/R the quotient set of U by the relation R, and

$$U/R = \{X_1, X_2, \ldots, X_m\}$$

where X_i is an equivalence class of $R, i = 1, 2, \ldots, m.$

Given an arbitrary set $X \in 2^U$, in general it may not be possible to describe X precisely in $\langle U, R\rangle$. One may characterize X by a pair of lower and upper approximations defined as follows [12]:

$$\underline{R}(X) = \bigcup_{X_i \subseteq X} X_i; \qquad \overline{R}(X) = \bigcup_{X_i \cap X \neq \emptyset} X_i$$

The pair $(\underline{R}(X), \overline{R}(X))$ is the representation of an ordinary set X in the approximation space $\langle U, R\rangle$ or simply called the rough set of X.

In [13], Pawlak introduces two numerical chacracterizations of imprecision of a subset X in the approximation space $\langle U, R\rangle$: *accuracy* and *roughness*. Accuracy of X, denoted by $\alpha_R(X)$, is simply the ratio of the number of objects in its lower approximation to that in its upper approximation; namely

$$\alpha_R(X) = \frac{|\underline{R}(X)|}{|\overline{R}(X)|} \tag{1}$$

where $|\cdot|$ denotes the cardinality of a set. Then the roughness of X, denoted by $\rho_R(X)$, is defined by subtracting the accuracy from 1:

$$\rho_R(X) = 1 - \alpha_R(X) = 1 - \frac{|\underline{R}(X)|}{|\overline{R}(X)|} \tag{2}$$

Note that the lower the roughness of a subset, the better is its approximation.

1) As $\underline{R}(X) \subseteq X \subseteq \overline{R}(X)$, $0 \leq \rho_R(X) \leq 1$.
2) By convention, when $X = \emptyset$, $\underline{R}(X) = \overline{R}(X) = \emptyset$ and $\rho_R(X) = 0$.
3) $\rho_R(X) = 0$ if and only if X is definable in $\langle U, R\rangle$.

In the rough set theory, the approximation quality γ is often used to describe the degree of partial dependency between attributes.

Assume now there is another equivalence relation P defined on U, which forms a partition (or, classification) U/P of U, say $U/P = \{Y_1, \ldots, Y_n\}$. Note that R and P may be induced respectively by sets of attributes applied to objects in U. Then the approximation quality of P by R, also called the *degree of dependency*, is defined by

$$\gamma_R(P) = \frac{\sum_{i=1}^{n} |\underline{R}(Y_i)|}{|U|} \tag{3}$$

which is represented in terms of accuracy as follows

$$\gamma_R(P) = \sum_{i=1}^{n} \frac{|\overline{R}(Y_i)|}{|U|} \alpha_R(Y_i) \tag{4}$$

In this case the measure $\gamma_R(P)$ can be regarded as the weighted mean of the accuracies of approximation of P by R [8].

B. Fuzzy Sets and Mass Assignment

Let U be a finite and non-empty set. A fuzzy set F of U is nothing but a mapping from U into the unit interval $[0, 1]$:

$$\mu_F : U \longrightarrow [0, 1]$$

where for each $x \in U$ we call $\mu_F(x)$ the membership degree of x in F. Practically, we may consider U as a set of objects of concern, and a crisp subset of U represent a "non-vague" concept imposed on objects in U. Then a fuzzy set F of U is thought of as a mathematical representation of a "vague" concept described linguistically.

Given a number $\alpha \in (0, 1]$, the α-cut, or α-level set, of F is defined as follows

$$F_\alpha = \{x \in U | \mu_F(x) \geq \alpha\}$$

which is a subset of U.

In connection to the evidence theory, a fuzzy set F is a consonant random set; the family of its α-cuts forms a nested family of focal elements [6]. Note that in this case the normalization assumption of F is imposed due to the body of evidence does not contain the empty set. Interestingly, this view of fuzzy sets has been used by Baldwin in [1], [2] to introduce the so-called mass assignment of a fuzzy set with relaxing the assumption, and to provide a probability based semantics for a fuzzy concept defined as a family of possible definitions of the concept. The mass assignment of a fuzzy set is defined as follows.

Let F be a fuzzy subset of a finite universe U such that the range of the membership function μ_F, denoted by rng(μ_F), is rng$(\mu_F) = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_i > \alpha_{i+1} > 0$, for $i = 1, \ldots, n-1$. Let

$$F_i = \{x \in U | \mu_F(x) \geq \alpha_i\}$$

for $i = 1, \ldots, n$. Then the mass assignment of F, denoted by m_F, is a probability distribution on 2^U definded by

$$\begin{aligned} m_F(\emptyset) &= 1 - \alpha_1 \\ m_F(F_i) &= \alpha_i - \alpha_{i+1}, \text{ for } i = 1, \ldots, n, \end{aligned}$$

with $\alpha_{n+1} = 0$ by convention. The α-level sets F_i, $i = 1, \ldots, n$, (or $\{F_i\}_{i=1}^n \cup \{\emptyset\}$ if F is a subnormal fuzzy set, i.e. $\max_{x \in U}\{\mu_F(x)\} < 1$) are referred to as the focal elements of m_F. The mass assignment of a fuzzy concept is then considered as providing a probabilistic based semantics for the membership function of the fuzzy concept.

III. Roughness Measures of a Fuzzy Set

A. Rough Fuzzy Sets

Let a finite approximation space $\langle U, R\rangle$ be given. Let F be a fuzzy set in U with the membership function μ_F. The upper and lower approximations $\overline{R}(F)$ and $\underline{R}(F)$ of F by R are fuzzy sets in the quotient set U/R with membership functions defined [7] by

$$\mu_{\overline{R}(F)}(X_i) = \max_{x \in X_i}\{\mu_F(x)\} \tag{5}$$

$$\mu_{\underline{R}(F)}(X_i) = \min_{x \in X_i}\{\mu_F(x)\} \tag{6}$$

for $i = 1, \ldots, m$. $(\underline{R}(F), \overline{R}(F))$ is called a rough fuzzy set.

The rough fuzzy set $(\underline{R}(F), \overline{R}(F))$ then induces two fuzzy sets F^* and F_* in U with membership functions defined respectively as follows

$$\mu_{F^*}(x) = \mu_{\overline{R}(F)}(X_i) \text{ and } \mu_{F_*}(x) = \mu_{\underline{R}(F)}(X_i)$$

if $x \in X_i$, for $i = 1, \ldots, m$. That is, F^* and F_* are fuzzy sets with constant membership degree on the equivalence classes of U by R, and for any $x \in U$, $\mu_{F^*}(x)$ (respectively, $\mu_{F_*}(x)$) can be viewed as the degree to which x possibly (respectively, definitely) belongs to the fuzzy set F [3].

Under such a view, we now define the notion of a *definable fuzzy set* in $\langle U, R\rangle$. A fuzzy set F is called *definable* if $\underline{R}(F) = \overline{R}(F)$, i.e. there exists a fuzzy set $\mathcal{F}$ in U/R such that $\mu_F(x) = \mu_{\mathcal{F}}(X_i)$ if $x \in X_i$, $i = 1 \ldots, m$. Further, as defined in [3], fuzzy sets F and G in U are said to be *roughly equal*, denoted by $F \approx_R G$, if and only if

$$\underline{R}(F) = \underline{R}(G) \text{ and } \overline{R}(F) = \overline{R}(G).$$

B. Roughness Measures of Fuzzy Sets

In [3], Banerjee and Pal have proposed a roughness measure for fuzzy sets in a given approximation space. Essentially, their measure of roughness of a fuzzy set depends on parameters that are designed as thresholds of definiteness and possibility in membership of the objects in U to the fuzzy set.

Consider parameters α, β such that $0 < \beta \leq \alpha \leq 1$. The α-cut $(F_*)_\alpha$ and β-cut $(F^*)_\beta$ of fuzzy sets F_* and F^*, respectively, are called to be the *α-lower approximation*, the *β-upper approximation* of F in $\langle U, R\rangle$, respectively. Then a roughness measure of the fuzzy set F with respect to parameters α, β with $0 < \beta \leq \alpha \leq 1$, and the approximation space $\langle U, R\rangle$ is defined by

$$\rho_R^{\alpha,\beta}(F) = 1 - \frac{|(F_*)_\alpha|}{|(F^*)_\beta|} \tag{7}$$

It is obvious that this definition of roughness measure $\rho_R^{\alpha,\beta}(\cdot)$ strongly depends on parameters α and β.

As pointed out in [9], this measure of roughness has several undesirable properties. Simultaneously, the authors also introduce a parameter-free measure of roughness of a fuzzy set as follows.

Let F be a normal fuzzy set in U. Assume that the range of the membership function μ_F is $\{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_i > \alpha_{i+1} > 0$, for $i = 1, \ldots, n-1$, and $\alpha_1 = 1$. Let us denote m_F the mass assignment of F defined as in the preceding section. Let

$$F_i = \{x \in U | \mu_F(x) \geq \alpha_i\}, \text{ for } i = 1, \ldots, n.$$

With these notations, the roughness measure of F with respect to the approximation space $\langle U, R\rangle$ is defined by

$$\hat{\rho}_R(F) = \sum_{i=1}^n m_F(F_i)(1 - \frac{|\underline{R}(F_i)|}{|\overline{R}(F_i)|}) \equiv \sum_{i=1}^n m_F(F_i)\rho_R(F_i) \tag{8}$$

That is, the roughness of a fuzzy set F is the weighted sum of the roughness measures of nested focal subsets which are considered as its possible definitions.

Observation 1:
- Clearly, $0 \leq \hat{\rho}_R(F) \leq 1$.
- $\hat{\rho}_R(\cdot)$ is a natural extension of Pawlak's roughness measure for fuzzy sets, i.e. if F is a crisp subset of U then $\hat{\rho}_R(F) = \rho_R(F)$.
- F is a definable fuzzy set if and only if $\hat{\rho}_R(F) = 0$.

Let F^* and F_* be fuzzy sets induced from the rough fuzzy set $(\underline{R}(F), \overline{R}(F))$ as above. Denote

$$\text{rng}(\mu_{F_*}) \cup \text{rng}(\mu_{F^*}) = \{\omega_1, \ldots, \omega_p\}$$

such that $\omega_i > \omega_{i+1} > 0$ for $i = 1, \ldots, p-1$. Obviously, $\{\omega_1, \ldots, \omega_p\} \subseteq \text{rng}(\mu_F)$, and $\omega_1 = \alpha_1$ and $\omega_p \geq \alpha_n$. With this notation, we have

Proposition 1: For any $1 \leq j \leq p$, if there exists $\alpha_i, \alpha_{i'} \in \text{rng}(\mu_F)$ such that $\omega_{j+1} < \alpha_i < \alpha_{i'} \leq \omega_j$ then we have $F_i \approx_R F_{i'}$,[1] and so $\rho_R(F_i) = \rho_R(F_{i'})$.

Further, we can represent the roughness $\hat{\rho}_R(F)$ in terms of level sets of fuzzy sets F_* and F^* in the following proposition.

Proposition 2: We have

$$\hat{\rho}_R(F) = \sum_{j=1}^p (\omega_j - \omega_{j+1})(1 - \frac{|(F_*)_{\omega_j}|}{|(F^*)_{\omega_j}|})$$

where $\omega_{p+1} = 0$, by convention.

More interestingly, we obtain the following.

Proposition 3: If fuzzy sets F and G in U are roughly equal in $\langle U, R\rangle$, then we have $\hat{\rho}_R(F) = \hat{\rho}_R(G)$.

IV. Rough Approximation Quality of a Fuzzy Classification

Recall that roughness of a crisp set is defined as opposed to its accuracy. First, in the following we will see that it is possible to make a similar correspondency between the roughness and accuracy of a fuzzy set.

It should be noticed that if F is a subnormal fuzzy set, we have $m_F(\emptyset) > 0$, and then the empty set may be also considered as a possible definition of F. In this case, we should define the roughness measure of F as

$$\hat{\rho}_R(F) = \sum_{i=1}^n m_F(F_i)\rho_R(F_i) + m_F(\emptyset)\rho_R(\emptyset) \tag{9}$$

which trivially turns back to the normal case above as, by convention, $\rho_R(\emptyset) = 0$. However, we should take the case into account when once we want to consider the accuracy measure instead of roughness, with the convention that $\alpha_R(\emptyset) = 1$.

Under such an observation, it is eligible to define the accuracy measure for a fuzzy set F by

$$\hat{\alpha}_R(F) = \sum_{i=1}^n m_F(F_i)\alpha_R(F_i) \tag{10}$$

[1] Note that F_i stands for F_{α_i}

if F is a normal fuzzy set, or

$$\hat{\alpha}_R(F) = \sum_{i=1}^{n} m_F(F_i)\alpha_R(F_i) + m_F(\emptyset)\alpha_R(\emptyset) \quad (11)$$

if F is a subnormal fuzzy set. With this definition we have

$$\hat{\alpha}_R(F) = 1 - \hat{\rho}_R(F) \quad (12)$$

for any fuzzy set F in U.

Before extending the the measure of rough dependency defined by (3) (or equivalently, (4)) for the case where P is a fuzzy classification of U instead of a crisp one, let us define the cardinality of a fuzzy set in the spirit of its probabilistic based semantics. That is, if $\{F_i\}_{i=1}^{n}$ could be interpreted as a family of possible definitions of the concept F, then $m_F(F_i)$ is the probability of the event "the concept is F_i", for each i. Under such an interpretation, the cardinality of F, also denoted by $|F|$, is defined as the expected cardinality by

$$|F| = \sum_{i=1}^{n} m_F(F_i)|F_i| \quad (13)$$

Quite interestingly, the following proposition shows that the expected cardinality (13) is nothing but the Σ-count of the fuzzy set F as introduced by De Luca and Termini [5].

Proposition 4: We have

$$|F| = \sum_{i=1}^{n} m_F(F_i)|F_i| = \sum_{x \in U} \mu_F(x) \quad (14)$$

Let us return to an approximation space $\langle U, R\rangle$ and assume further a fuzzy partition, say $\mathcal{FC} = \{Y_1, \ldots, Y_k\}$, defined on U. This situation may come up in a natural way when a linguistic classification is defined on U and must be approximated in terms of already existing knowledge R.

In such a situation, quite naturally with the spirit of the proposal described in the preceding section, one may define the approximation quality of $\mathcal{FC}$ by R as

$$\hat{\gamma}_R(\mathcal{FC}) = \frac{1}{|U|}\sum_{i=1}^{k}\sum_{j=1}^{n_i} m_{Y_i}(Y_{i,j})|\underline{R}(Y_{i,j})| \quad (15)$$

where for $i = 1, \ldots, k$, m_{Y_i} and $\{Y_{i,j}\}_{j=1}^{n_i}$ respectively stand for the mass assignment of Y_i and the family of its focal elements. Straightforwardly, it follows from Proposition 4 that

$$\hat{\gamma}_R(\mathcal{FC}) = \frac{1}{|U|}\sum_{i=1}^{k} |(Y_i)_*| \quad (16)$$

where $(Y_i)_*$, $i = 1, \ldots, k$, are fuzzy sets with constant membership degree on the equivalence classes of U by R as defined in Section III. It is also interesting to note that the approximation quality of $\mathcal{FC}$ by R can be also extended via (4) as follows

$$\hat{\gamma}'_R(\mathcal{FC}) = \sum_{i=1}^{k} \frac{|\overline{R}(Y_i)|}{|U|}\hat{\alpha}_R(Y_i) \quad (17)$$

However, we will not consider this extension in the rest of the paper.

Furthermore, similar as mentioned in [13], the measure of rough dependency $\hat{\gamma}_R$ does not capture how this partial dependency is actually distributed among fuzzy classes of $\mathcal{FC}$. To capture this information we need also the so-called *precision measure* $\hat{\pi}_R(Y_i)$, for $i = 1, \ldots, k$, defined by

$$\hat{\pi}_R(Y_i) = \sum_{j=1}^{n_i} m_{Y_i}(Y_{i,j})\frac{|\underline{R}(Y_{i,j})|}{|Y_{i,j}|} \quad (18)$$

which may be considered as the expected relative number of elements in Y_i approximated by R. Clearly, we have $\hat{\pi}_R(Y_i) \geq \hat{\alpha}_R(Y_i)$, for any $i = 1, \ldots, k$. As such the two measures $\hat{\gamma}_R$ and $\hat{\pi}_R$ give us enough information about the "classification power" of the knowledge R with respect to the linguistic classification $\mathcal{FC}$.

In rough-set-based data analysis, R is naturally induced by a subset, say B, of the set of attributes imposed on objects being considered. Then as suggested in [13], we can also measure the significance of the subset of attributes $B' \subseteq B$ with respect to the linguistic classification $\mathcal{FC}$ by the difference

$$\hat{\gamma}_R(\mathcal{FC}) - \hat{\gamma}_{R'}(\mathcal{FC})$$

where R' denotes the equivalence relation induced by the subset of attributes $B \setminus B'$. This measure expresses how influence on the quality of approximation if we drop attributes in B' from B.

For the sake of illustration, in the following section we will consider a simple example depicting the introduced notions.

V. An Example

Let us consider a relation in a relational database as shown in Table I (this database is a variant of that found in [4]). Then by the attributes **Degree** and **Experience** we obtain an approximation space

$$\langle U, \text{ind}(\{\textbf{Degree}, \textbf{Experience}\})\rangle$$

where $U = \{1, \ldots, 16\}$, and the corresponding partition as shown at the top of the page. Further, consider now for example a linguistic classification

$$\{Low, Medium, High\}$$

defined on the domain of attribute **Salary**, say [20K,70K], with membership functions of linguistic classes depicted graphically as in Fig. 1. Then the linguistic classification induces a fuzzy partition on U whose membership functions of fuzzy classes shown in Table II.

Then approximations of the fuzzy partition induced by **Salary** in the approximation space defined by **Degree** and **Experience** are given in Table III. Using (16) we obtain

$$\hat{\gamma}_{\{\textbf{Degree},\textbf{Experience}\}}(\textbf{Salary}) = \frac{13.46}{16} = 0.84$$

$$U/\text{ind}(\{\textbf{Degree}, \textbf{Experience}\}) = \{\{1,15\},\{2,6\},\{3,11,13,14\},\{4,12\},\{5,7\},\{8,9\},\{10,16\}\}$$

TABLE III
THE APPROXIMATIONS OF THE FUZZY PARTITION BASED ON **Salary**

X_i	$\{1,15\}$	$\{2,6\}$	$\{3,11,13,14\}$	$\{4,12\}$	$\{5,7\}$	$\{8,9\}$	$\{10,16\}$
μ_{High_*}	1	0.13	0.33	0	0	0	1
μ_{High^*}	1	0.33	0.67	0	0	0	1
μ_{Medium_*}	0	0.67	0.33	0	0	0.67	0
μ_{Medium^*}	0	0.87	0.67	0	0.33	0.73	0
μ_{Low_*}	0	0	0	1	0.67	0.27	0
μ_{Low^*}	0	0	0	1	1	0.33	0

TABLE I
RELATION IN A RELATIONAL DATABASE

ID	Degree	Experience (n)	Salary
1	Ph.D.	$6 < n \leq 8$	63K
2	Ph.D.	$0 < n \leq 2$	47K
3	M.S.	$6 < n \leq 8$	53K
4	B.S.	$0 < n \leq 2$	26K
5	B.S.	$2 < n \leq 4$	29K
6	Ph.D.	$0 < n \leq 2$	50K
7	B.S.	$2 < n \leq 4$	35K
8	M.S.	$2 < n \leq 4$	40K
9	M.S.	$2 < n \leq 4$	41K
10	M.S.	$8 < n \leq 10$	68K
11	M.S.	$6 < n \leq 8$	50K
12	B.S.	$0 < n \leq 2$	23K
13	M.S.	$6 < n \leq 8$	55K
14	M.S.	$6 < n \leq 8$	51K
15	Ph.D.	$6 < n \leq 8$	65K
16	M.S.	$8 < n \leq 10$	64K

TABLE II
INDUCED FUZZY PARTITION OF U BASED ON **Salary**

U	μ_{Low}	μ_{Medium}	μ_{High}
1	0	0	1
2	0	0.87	0.13
3	0	0.47	0.53
4	1	0	0
5	1	0	0
6	0	0.67	0.33
7	0.67	0.33	0
8	0.33	0.67	0
9	0.27	0.73	0
10	0	0	1
11	0	0.67	0.33
12	1	0	0
13	0	0.33	0.67
14	0	0.6	0.4
15	0	0	1
16	0	0	1

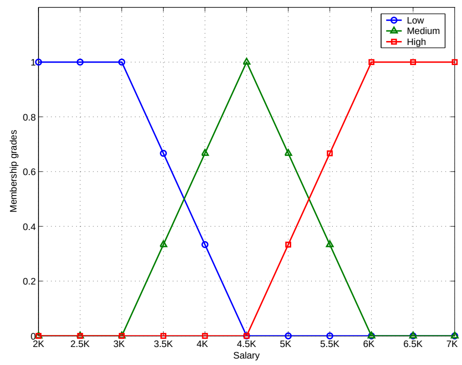

Fig. 1. A Linguistic Partition of **Salary** Attribute

That is we have the following partial dependency in the database

$$\{\textbf{Degree}, \textbf{Experience}\} \Rightarrow_{0.84} \textbf{Salary} \qquad (19)$$

To calculate the precision measure of fuzzy classes we need to obtain the mass assignment for each fuzzy class and approximations of its focal sets respectively. For example, the mass assignment of Low and approximations of its focal sets are shown in Table IV. Then we have

$$\hat{\pi}_{\{\textbf{Degree},\textbf{Experience}\}}(Low) = 0.878$$

Similarly, we also obtain

$$\hat{\pi}_{\{\textbf{Degree},\textbf{Experience}\}}(Medium) = 0.646$$
$$\hat{\pi}_{\{\textbf{Degree},\textbf{Experience}\}}(High) = 0.876$$

Now in order to show how the influence of, for example, attribute **Experience** on the quality of approximation, let us consider the partition induced by the attribute **Degree** as shown on the next page.

Then we obtain approximations of the fuzzy partition induced by **Salary** in the approximation space defined by **Degree** given in Table V. Thus we have

$$\hat{\gamma}_{\{\textbf{Degree}\}}(\textbf{Salary}) = \frac{3.2}{16} = 0.2$$

TABLE IV

MASS ASSIGNMENT FOR μ_{Low} AND APPROXIMATIONS OF ITS FOCAL SETS

α	1	0.67	0.33	0.27
Low_α	$\{4,5,12\}$	$\{4,5,12,7\}$	$\{4,5,12,7,8\}$	$\{4,5,12,7,8,9\}$
$m_{Low}(Low_\alpha)$	0.33	0.34	0.06	0.27
$\underline{R}(Low_\alpha)$	$\{4,12\}$	$\{4,5,12,7\}$	$\{4,5,12,7\}$	$\{4,5,12,7,8,9\}$
$\overline{R}(Low_\alpha)$	$\{4,5,12,7\}$	$\{4,5,12,7\}$	$\{4,5,12,7,8,9\}$	$\{4,5,12,7,8,9\}$

$$U/\text{ind}(\{\textbf{Degree}\}) = \{\{1,2,6,15\},\{3,8,9,10,11,13,14,16\},\{4,5,7,12\}\}$$

TABLE V

THE APPROXIMATIONS OF THE FUZZY PARTITION BASED ON **Salary**

X_i	$\{1,2,6,15\}$	$\{3,8,9,10,11,13,14,16\}$	$\{4,5,7,12\}$
μ_{High_*}	0.13	0	0
μ_{High^*}	1	1	0
μ_{Medium_*}	0	0	0
μ_{Medium^*}	0.87	0.73	0.33
μ_{Low_*}	0	0	0.67
μ_{Low^*}	0	0.33	1

Similarly, we also easily obtain

$$\hat{\gamma}_{\{\textbf{Experience}\}}(\textbf{Salary}) = \frac{5.06}{16} = 0.316$$

As we can see, both attributes **Degree** and **Experience** are highly significant as without each of them the measure of approximation quality changes considerably. It would be worth noting that based on background knowledge one may infer a dependency between $\{$**Degree**, **Experience**$\}$ and **Salary** which is often expressed linguistically, however such a dependency in general can not be described by traditional data dependencies.

VI. CONCLUSIONS

In this paper we have extended the measure of rough dependency for fuzzy classification for dealing with practical situations where a fuzzy classification must be approximated by available knowledge expressed in terms of a classical approximation space. Such situations may come up naturally for example when we want to realize partial dependency between attributes which is infered based on background knowledge; while such a dependency can not be expressed in terms of traditional data dependencies as described in Example.

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