A Balanced Model Reduction for Nonlinear Uncertain Systems via a T-S Fuzzy Approach

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Abstract- This paper deals with a balanced model reduction for a class of nonlinear systems with uncertain time varying parameters using a T-S(Takagi-Sugeno) fuzzy approach. We define a generalized controllability gramian and a generalized observability gramian for a stable T-S fuzzy systems with uncertainties. We obtain a balanced state space realization using the generalized controllability and observability gramian and obtain a reduced model by truncating not only states but also time varying uncertain parameters from the balanced state space realization. We also present an upper bound of the approximation error. The generalized controllability gramian and observability gramian can be computed from solutions of linear matrix inequalities. We demonstrate the efficacy of the suggested method by illustrating a numerical example.

I. Introduction

For linear finite dimensional systems with high orders, optimal control techniques such as linear quadratic gaussian and H_∞ control theory, usually produce controllers with the same state dimension as the model. Lower dimensional linear controllers are normally preferred over higher dimensional controllers in control system designs for some obvious reasons : they are easier to understand, implement and have higher reliability. Accordingly the problem of model reduction is of significant practical importance in control system design and has been a focus of a wide variety of studies for recent decades(see [1-4] and the references therein).

Model reduction techniques have been developed for linear uncertain systems as well. Using LMI machinery, Beck suggested a balanced truncation method for linear uncertain discrete systems with related error bounds[5], Model reduction techniques for linear parameter varying systems were also reported by several researchers [6,7,8]. However, comparatively little work has been reported for the model reduction for nonlinear systems.

In recent years, a controller design method for nonlinear dynamic systems modeled as a T-S(Takagi-Sugeno) fuzzy model has been intensively addressed[9,10,11]. Unlike a single conventional model, this T-S fuzzy model usually consists of several linear models to describe the global behavior of the nonlinear system. Typically the T-S fuzzy model is described by fuzzy IF-THEN rules. Based on this fuzzy model, many researchers use one of control design methods developed for linear parameter varying system. In order to alleviate computational burden in design phase and simplify the designed fuzzy controller, the state dimension of the T-S fuzzy model should be low.

In this paper, using a fuzzy approach we develop a balanced model reduction scheme for T-S fuzzy systems with norm bounded time varying uncertain parameters. In section II, we define the T-S fuzzy system with time varying uncertain parameters. A generalized controllability gramian and a generalized observability gramian are defined and a balanced realization of T-S fuzzy system using the generalized controllability and observability gramian is also presented in section III. A model approximation bound is derived and a suboptimal procedure is described to get a less conservative error bound in section IV. Section V demonstrates a numerical example and finally some concluding remarks are given in section VI.

The notation in this paper is fairly standard. R^n denotes n dimensional real vector space and $R^{n\times m}$ is the set of real $n\times m$ matrices. A^T denotes the transpose of a real matrix A. 0 and I denote zero matrix and identity matrix respectively. $M>0$ means that M is a positive definite matrix. In a block symmetric matrix, * in (i,j) block means the transpose of the matrix in (j,i) block. Finally $\|\cdot\|_\infty$ denotes the H_∞ norm of the system.

II. T-S Fuzzy System

We consider the following fuzzy dynamic system with uncertain time varying parameters.

Plant Rule i ($i=1,\cdots,r$) :
IF $\rho_1(t)$ is M_{i1} and $\cdots$ and $\rho_g(t)$ is M_{ig},
THEN

$$\begin{aligned}\dot{x}(t) &= A_i x(t) + F_i w(t) + B_i u(t)\\ z(t) &= H_i x(t) + J_i u(t)\\ y(t) &= C_i x(t) + G_i w(t) + D_i u(t)\\ w(t) &= \Theta(t) z(t)\end{aligned} \tag{1}$$

where r is the number of fuzzy rules. $\rho_j(t)$ and M_{ij} ($j = 1,\cdots,g$) are the premise variables and the fuzzy set respectively. $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input, $y(t) \in R^p$ is the output variable and $w(t) \in R^q, z(t) \in R^q$ are variables related to uncertain parameters. $A_i, F_{i,} \cdots, D_i$ are real matrices with compatible dimensions. $\Theta(t)$ is an uncertain parameter matrix defined as follows:

$$\Theta(t) = diag(\theta_1(t) I_{q1}, \theta_2(t) I_{q2}, \cdots, \theta_s(t) I_{qs}), \tag{2}$$
$$\Theta^T(t)\Theta(t) \le I\,, \quad q = q_1 + q_2 + \cdots + q_s\,.$$

Let $\mu_i(\rho(t))$, $i = 1,\cdots,r$,be the normalized membership function of the inferred fuzzy set $h_i(\rho(t))$,

$$\mu_i(\rho(t)) = \frac{h_i(\rho(t))}{\sum_{i=1}^{r} h_i(\rho(t))} \tag{3}$$

where

$$h_i(\rho(t)) = \prod_{j=1}^{g} M_{ij}(\rho_j(t))\,,$$

$$\rho(t) = \begin{bmatrix}\rho_1(t) & \rho_2(t) & \cdots & \rho_g(t)\end{bmatrix}^T. \tag{4}$$

In this paper, assuming for all i , $h_i(\rho(t)) \ge 0$ and $\sum_{i=1}^{r} h_i(\rho(t)) > 0$ we obtain

$$\mu_i(\rho(t)) \ge 0, \quad \sum_{i=1}^{r} \mu_i(\rho(t)) = 1\,. \tag{5}$$

For simplicity, by defining $\mu_i = \mu_i(\rho(t))$ and $\mu^T = \begin{bmatrix}\mu_1 & \cdots & \mu_r\end{bmatrix}$ the uncertain fuzzy system (1) can be written as follows :

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r} \mu_i (A_i x(t) + F_i w(t) + B_i u(t))\\ &= Ax(t) + F(\mu) w(t) + B(\mu) u(t)\\ z(t) &= \sum_{i=1}^{r} \mu_i (H_i x(t) + J_i u(t))\\ &= H(\mu) x(t) + J(\mu) u(t)\\ y(t) &= \sum_{i=1}^{r} \mu_i (C_i x(t) + G_i w(t) + D_i u(t))\\ &= C(\mu) x(t) + G(\mu) w(t) + D(\mu) u(t)\\ w(t) &= \Theta(t) z(t)\end{aligned} \tag{6}$$

In a packed matrix notation, we express the fuzzy system (6) as follows :

$$G_\Theta = \left[\begin{array}{c|c:c} A(\mu) & F(\mu) & B(\mu)\\ \hline H(\mu) & 0 & J(\mu)\\ \hdashline C(\mu) & G(\mu) & D(\mu)\end{array}\right] \tag{7}$$

III. A Balanced Realization

In this section we present a balanced realization for the uncertain fuzzy system (7) using generalized controllability and observability gramians. First we define generalized controllability and observability gramians.

Lemma 1 : (Generalized controllability and observability gramians)

1) Suppose that there exist $Q = Q^T > 0$ and $R = R^T > 0,$ $R = diag(R_1,\cdots,R_s),$ $R_i \in R^{q_i \times q_i}\,(i = 1,\cdots,s)$ satisfying LMI (9), then the output energy is bounded above as follows :

$$\int_0^\infty y(t)^T y(t) dt < x(0)^T Q x(0) \quad \text{for} \quad u(t) \equiv 0\,. \tag{8}$$

$$L_{oi} = \begin{bmatrix} A_i^T Q + Q A_i & * & * & * \\ R H_i & -R & * & * \\ F_i^T Q & 0 & -R & * \\ C_i & 0 & G_i & -I \end{bmatrix} < 0, i = 1,\cdots,r \tag{9}$$

2) Suppose that there exist $P = P^T > 0$ and $S = S^T > 0,$ $S = diag(S_1,\cdots,S_s),$ $S_i \in R^{q_i \times q_i}\,(i = 1,\cdots,s)$ satisfying LMI (11), the input energy transferring from $x(-\infty) = 0$ to $x(0) = x_0$ is bounded below as follows :

$$\int_{-\infty}^0 u(t)^T u(t) dt > x_0{}^T P^{-1} x_0 \tag{10}$$

$$L_{ci} = \begin{bmatrix} P A_i^T + A_i P & * & * & * \\ H_i P & -S & * & * \\ S F_i^T & 0 & -S & * \\ B_i^T & J_i^T & 0 & -I \end{bmatrix} < 0, i = 1,\cdots,r \tag{11}$$

(proof) The proof is omitted due to space limitation.

As in [6], we say Q and P ,solutions of LMI's (9) and (11), are generalized observability gramian and controllability gramian respectively. While the observability and controllability gramian in linear time invariant systems are unique, the generalized gramians of the fuzzy system (7) are not unique. But the generalized gramians are related to the input and output energy as can be seen in lemma 1.

Using the generalized gramians, we suggest a balanced realization of the uncertain fuzzy system (7). We obtain a transformation matrix T and W satisfying

$$\Sigma = diag(\sigma_1,\sigma_2,\cdots,\sigma_n) = T^T Q T = T^{-1} P T^{-T}$$
$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n$$

$$\begin{aligned}\Pi &= diag(\Phi_1,\Phi_2,\cdots,\Phi_s) = diag(\pi_1,\pi_2,\cdots,\pi_q) \\ &= W^T R W = W^{-1} S W^{-T} \\ & tr(\Phi_1) \ge tr(\Phi_2) \ge \cdots \ge tr(\Phi_s)\end{aligned} \tag{12}$$

With T and W defined in (12), the change of coordinates in the fuzzy system (7) gives

$$G_{\Theta_b} = \left[\begin{array}{c|c:c} A_b(\mu) & F_b(\mu) & B_b(\mu) \\ \hline H_b(\mu) & 0 & J_b(\mu) \\ \hdashline C_b(\mu) & G_b(\mu) & D_b(\mu) \end{array}\right]$$
$$= \left[\begin{array}{c|c:c} T^{-1}A(\mu)T & T^{-1}F(\mu)W & T^{-1}B(\mu) \\ \hline W^{-1}H(\mu)T & 0 & W^{-1}J(\mu) \\ \hdashline C(\mu)T & G(\mu)W & D(\mu) \end{array}\right] \tag{13}$$

where $\Theta_b(t) = W^{-1}\Theta(t)W$.

One can easily observe that the state space realization of (13) satisfy following LMI's (14) and (15).

$$L_o(\mu) = \begin{bmatrix} A_b(\mu)^T\Sigma + \Sigma A_b(\mu) + C_b(\mu)^T C_b(\mu) & * & * \\ \Pi H_b(\mu) & -\Pi & * \\ G_b(\mu)^T C_b(\mu) + F_b(\mu)^T\Sigma & 0 & G_b(\mu)^T G_b(\mu) - \Pi \end{bmatrix} < 0 \tag{14}$$

$$L_c(\mu) = \begin{bmatrix} \Sigma A_b(\mu)^T + A_b(\mu)\Sigma + B_b(\mu)B_b(\mu)^T & * & * \\ H_b(\mu)\Sigma + J_b(\mu)B_b(\mu)^T & J_b(\mu)J_b(\mu)^T - \Pi & * \\ \Pi F_b(\mu)^T & 0 & -\Pi \end{bmatrix} < 0 \tag{15}$$

From this reason, we say that the realization (13) is a balanced realization of the fuzzy system (7) and Σ is a balanced gramian.

IV. Balanced Model Reduction

In this section, we develop a balanced model reduction scheme using the balanced gramian defined in section III. We also derive an upper bound of model approximation error. We assume that the fuzzy system (7) is already balanced and partitioned as follows :

$$G_\Theta = \left[\begin{array}{cc|cc:c} A_{11}(\mu) & A_{12}(\mu) & F_{11}(\mu) & F_{12}(\mu) & B_1(\mu) \\ A_{21}(\mu) & A_{22}(\mu) & F_{21}(\mu) & F_{22}(\mu) & B_2(\mu) \\ \hline H_{11}(\mu) & H_{12}(\mu) & 0 & 0 & J_1(\mu) \\ H_{21}(\mu) & H_{22}(\mu) & 0 & 0 & J_2(\mu) \\ \hdashline C_1(\mu) & C_2(\mu) & G_1(\mu) & G_2(\mu) & D(\mu) \end{array}\right]$$

$$= \sum_{i=1}^{r} \mu_i \left[\begin{array}{cc|cc:c} A_{i,11} & A_{i,12} & F_{i,11} & F_{i,12} & B_{i,1} \\ A_{i,21} & A_{i,22} & F_{i,21} & F_{i,22} & B_{i,2} \\ \hline H_{i,11} & H_{i,12} & 0 & 0 & J_{i,1} \\ H_{i,21} & H_{i,22} & 0 & 0 & J_{i,2} \\ \hdashline C_{i,1} & C_{i,2} & G_{i,1} & G_{i,2} & D_i \end{array}\right] \tag{16}$$
$$w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} \Theta_1(t) & 0 \\ 0 & \Theta_2(t) \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

where $A_{11}(\mu) \in R^{k\times k}$, and the other matrices are compatibly partitioned.

From (16) we obtain a reduced order model by truncating $n-k$ states and $q-v$ uncertain parameters as follows :

$$\bar{G}_{\Theta_1} = \left[\begin{array}{c|c:c} A_{11}(\mu) & F_{11}(\mu) & B_1(\mu) \\ \hline H_{11}(\mu) & 0 & J_1(\mu) \\ \hdashline C_1(\mu) & G_1(\mu) & D(\mu) \end{array}\right] = \sum_{i=1}^{r} \mu_i \left[\begin{array}{c|c:c} A_{i,11} & F_{i,11} & B_{i,1} \\ \hline H_{i,11} & 0 & J_{i,1} \\ \hdashline C_{i,1} & G_{i,1} & D_i \end{array}\right] \tag{17}$$
$$w_1(t) = \Theta_1(t) z_1(t)$$

Theorem 2 : The reduced order system (17) is quadratically stable and balanced. Moreover the model approximation error is given by

$$\left\| G_\Theta - \bar{G}_{\Theta_1} \right\|_\infty \le 2\left(\sum_{j=k+1}^{n} \sigma_j + \sum_{j=v+1}^{q} \pi_j\right) \tag{18}$$

(proof) We partition $\Sigma = diag(\Sigma_1, \Sigma_2)$ and $\Pi = diag(\Pi_1, \Pi_2)$ where $\Sigma_1 \in R^{k\times k}$, $\Pi_1 \in R^{v\times v}$. Then the reduced order system (17) satisfies LMI's (19) and (20).

$$\begin{bmatrix} A_{11}(\mu)^T\Sigma_1 + \Sigma_1 A_{11}(\mu) + C_1(\mu)^T C_1(\mu) & * & * \\ \Pi_1 H_{11}(\mu) & -\Pi_1 & * \\ G_1(\mu)^T C_1(\mu) + F_{11}(\mu)^T\Sigma_1 & 0 & G_1(\mu)^T G_1(\mu) - \Pi_1 \end{bmatrix} < 0 \tag{19}$$

$$\begin{bmatrix} \Sigma_1 A_{11}(\mu)^T + A_{11}(\mu)\Sigma_1 + B_1(\mu)B_1(\mu)^T & * & * \\ H_{11}(\mu)\Sigma_1 + J_1(\mu)B_1(\mu)^T & J_1(\mu)J_1(\mu)^T - \Pi_1 & * \\ \Pi_1 F_{11}(\mu)^T & 0 & -\Pi_1 \end{bmatrix} < 0 \tag{20}$$

Hence the reduced order system is quadratically stable and balanced. Without loss of generality we consider 2 cases.
Case1 : ($k = n-1,\ \ v = q$)

Note that in this case $F_{12}(\mu)$, $F_{22}(\mu)$, $H_{21}(\mu)$, $H_{22}(\mu)$, $J_2(\mu)$ and $G_2(\mu)$ are empty matrices. Hence a state space realization of the error system $G^e_{\Theta_e} = G_\Theta - \bar{G}_{\Theta_1}$ can be written by

$$G^e_{\Theta_e} = \left[\begin{array}{c|c:c} \bar{A}_e(\mu) & \bar{F}_e(\mu) & \bar{B}_e(\mu) \\ \hline \bar{H}_e(\mu) & 0 & \bar{J}_e(\mu) \\ \hdashline \bar{C}_e(\mu) & \bar{G}_e(\mu) & 0 \end{array}\right] = \left[\begin{array}{ccc|cc:c} A_{11}(\mu) & 0 & 0 & F_{11}(\mu) & 0 & B_1(\mu) \\ 0 & A_{11}(\mu) & A_{12}(\mu) & 0 & F_{11}(\mu) & B_1(\mu) \\ 0 & A_{21}(\mu) & A_{22}(\mu) & 0 & F_{21}(\mu) & B_2(\mu) \\ \hline H_{11}(\mu) & 0 & 0 & 0 & 0 & J(\mu) \\ 0 & H_{11}(\mu) & H_{12}(\mu) & 0 & 0 & J(\mu) \\ \hdashline -C_1(\mu) & C_1(\mu) & C_2(\mu) & -G(\mu) & G(\mu) & 0 \end{array}\right] \tag{21}$$

where $\Theta_e(t) = diag(\Theta(t), \Theta(t))$.

The change of coordinate in the error system gives

$$G^e_{\Theta_e} = \left[\begin{array}{c|c:c} A_e(\mu) & F_e(\mu) & B_e(\mu) \\ \hline H_e(\mu) & 0 & J_e(\mu) \\ \hdashline C_e(\mu) & G_e(\mu) & 0 \end{array}\right] = \left[\begin{array}{c|c:c} M^{-1}\bar{A}_e(\mu)M & M^{-1}\bar{F}_e(\mu) & M^{-1}\bar{B}_e(\mu) \\ \hline \bar{H}_e(\mu)M & 0 & \bar{J}_e(\mu) \\ \hdashline \bar{C}_e(\mu)M & \bar{G}_e(\mu) & 0 \end{array}\right], \tag{22}$$

where

$$M = \begin{bmatrix} I & I & 0 \\ I & -I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad B_e(\mu) = \begin{bmatrix} B_1(\mu) \\ 0 \\ B_2(\mu) \end{bmatrix},$$

$$A_e(\mu) = \begin{bmatrix} A_{11}(\mu) & 0 & A_{12}(\mu)/2 \\ 0 & A_{11}(\mu) & -A_{12}(\mu)/2 \\ A_{21}(\mu) & -A_{21}(\mu) & A_{22}(\mu) \end{bmatrix},$$

$$F_e(\mu) = \begin{bmatrix} F_{11}(\mu)/2 & F_{11}(\mu)/2 \\ F_{11}(\mu)/2 & -F_{11}(\mu)/2 \\ 0 & F_{21}(\mu) \end{bmatrix},$$

$$H_e(\mu) = \begin{bmatrix} H_{11}(\mu) & H_{11}(\mu) & 0 \\ H_{11}(\mu) & -H_{11}(\mu) & H_{12}(\mu) \end{bmatrix},$$

$$J_e(\mu) = \begin{bmatrix} J(\mu) \\ J(\mu) \end{bmatrix}, \quad C_e(\mu) = \begin{bmatrix} 0 & -2C_1(\mu) & C_2(\mu) \end{bmatrix},$$

$$G_e(\mu) = \begin{bmatrix} -G(\mu) & G(\mu) \end{bmatrix}.$$

It is well known that the existence of $\Sigma_e = \Sigma_e^T > 0$ and $\Pi_e = \Pi_e^T > 0$ $\Pi_e \Theta_e(t) = \Theta_e(t)\Pi_e$ satisfying following LMI (23) guarantees $\left\| G^e_{\Theta_e} \right\|_\infty \le \gamma$.

$$\begin{bmatrix} \Gamma_{11} & * & * \\ H_e(\mu)\Sigma_e + J_e(\mu)B_e(\mu)^T & J_e(\mu)J_e(\mu)^T - \Pi_e & * \\ \Pi_e F_e(\mu)^T + \gamma^{-2}\Pi_e G_e(\mu)^T C_e(\mu)\Sigma_e & 0 & \Gamma_{33} \end{bmatrix} < 0, \tag{23}$$

where

$$\Gamma_{11} = \Sigma_e A_e(\mu)^T + A_e(\mu)\Sigma_e + B_e(\mu)B_e(\mu)^T + \gamma^{-2}\Sigma_e C_e(\mu)^T C_e(\mu)\Sigma_e$$
$$\Gamma_{33} = \gamma^{-2}\Pi_e G_e(\mu)^T G_e(\mu)\Pi_e - \Pi_e.$$

Let $\gamma = 2\sigma_n$, $\Sigma_e = diag(\Sigma_1, \sigma_n^2\Sigma_1^{-1}, 2\sigma_n)$ and $\Pi_e = \begin{bmatrix} \Pi + \sigma_n^2\Pi^{-1} & \Pi - \sigma_n^2\Pi^{-1} \\ \Pi - \sigma_n^2\Pi^{-1} & \Pi + \sigma_n^2\Pi^{-1} \end{bmatrix}$. Then LMI (23) can be expressed as follows :

$$L = \begin{bmatrix} U_1^T & 0 & 0 \\ 0 & U_2^T & 0 \\ 0 & 0 & U_2^T \end{bmatrix} L_c(\mu) \begin{bmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_2 \end{bmatrix} + \begin{bmatrix} V_1^T & 0 & 0 \\ 0 & V_2^T & 0 \\ 0 & 0 & V_2^T \end{bmatrix} L_o(\mu) \begin{bmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_2 \end{bmatrix} < 0 \tag{24}$$

where

$$U_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad U_2^T = \begin{bmatrix} I \\ I \end{bmatrix},$$
$$V_1 = \begin{bmatrix} 0 & \sigma_n\Sigma_1^{-1} & 0 \\ 0 & 0 & -I \end{bmatrix}, \quad V_2^T = \begin{bmatrix} \sigma_n\Pi^{-1} \\ -\sigma_n\Pi^{-1} \end{bmatrix}.$$

Case 2 : ($k = n$, $v = q-1$)

In this case, $A_{12}(\mu)$, $A_{21}(\mu)$, $A_{22}(\mu)$, $F_{21}(\mu)$, $F_{22}(\mu)$, $H_{12}(\mu)$ and $H_{22}(\mu)$ are empty matrices so that the error system becomes

$$G^e_{\Theta_e} = \left[\begin{array}{c|c:c} \bar{A}_e(\mu) & \bar{F}_e(\mu) & \bar{B}_e(\mu) \\ \hline \bar{H}_e(\mu) & 0 & \bar{J}_e(\mu) \\ \hdashline \bar{C}_e(\mu) & \bar{G}_e(\mu) & 0 \end{array}\right] = \left[\begin{array}{cc|ccc:c} A(\mu) & 0 & F_{11}(\mu) & 0 & 0 & B(\mu) \\ 0 & A(\mu) & 0 & F_{11}(\mu) & F_{12}(\mu) & B(\mu) \\ \hline H_{11}(\mu) & 0 & 0 & 0 & 0 & J_1(\mu) \\ 0 & H_{11}(\mu) & 0 & 0 & 0 & J_1(\mu) \\ 0 & H_{21}(\mu) & 0 & 0 & 0 & J_2(\mu) \\ \hdashline -C(\mu) & C(\mu) & -G_1(\mu) & G_1(\mu) & G_2(\mu) & 0 \end{array}\right], \tag{25}$$

where $\Theta_e(t) = diag(\Theta_1(t), \Theta(t))$. The change of coordinate in the error system gives

$$
\begin{aligned}
G^e_{\Theta_e} &= \left[\begin{array}{c|c:c} A_e(\mu) & F_e(\mu) & B_e(\mu) \\ \hline H_e(\mu) & 0 & J_e(\mu) \\ \hdashline C_e(\mu) & G_e(\mu) & 0 \end{array}\right] \\
&= \left[\begin{array}{c|c:c} M^{-1}\bar{A}_e(\mu)M & M^{-1}\bar{F}_e(\mu) & M^{-1}\bar{B}_e(\mu) \\ \hline \bar{H}_e(\mu)M & 0 & \bar{J}_e(\mu) \\ \hdashline \bar{C}_e(\mu)M & \bar{G}_e(\mu) & 0 \end{array}\right],
\end{aligned} \tag{26}
$$

where

$$
M = \begin{bmatrix} I & I \\ I & -I \end{bmatrix},\ A_e(\mu) = \begin{bmatrix} A(\mu) & 0 \\ 0 & A(\mu) \end{bmatrix}, B_e(\mu) = \begin{bmatrix} B(\mu) \\ 0 \end{bmatrix},
$$

$$
F_e(\mu) = \begin{bmatrix} F_{11}(\mu)/2 & F_{11}(\mu)/2 & F_{12}(\mu)/2 \\ F_{11}(\mu)/2 & -F_{11}(\mu)/2 & -F_{12}(\mu)/2 \end{bmatrix},
$$

$$
H_e(\mu) = \begin{bmatrix} H_{11}(\mu) & H_{11}(\mu) \\ H_{11}(\mu) & -H_{11}(\mu) \\ H_{21}(\mu) & -H_{21}(\mu) \end{bmatrix}, J_e(\mu) = \begin{bmatrix} J_1(\mu) \\ J_1(\mu) \\ J_2(\mu) \end{bmatrix},
$$

$$
C_e(\mu) = \begin{bmatrix} 0 & -2C(\mu) \end{bmatrix},
$$

$$
G_e(\mu) = \begin{bmatrix} -G_1(\mu) & G_1(\mu) & G_2(\mu) \end{bmatrix}.
$$

We define $\gamma = 2\pi_q$ and

$$
\Pi = \begin{bmatrix} \Pi_1 & 0 \\ 0 & \pi_q \end{bmatrix},\ \Sigma_e = \begin{bmatrix} \Sigma & 0 \\ 0 & \pi_q^2\Sigma^{-1} \end{bmatrix},
$$

$$
\Pi_e = \begin{bmatrix} \Pi_1 + \pi_q^2\Pi_1^{-1} & \Pi_1 - \pi_q^2\Pi_1^{-1} & 0 \\ \Pi_1 - \pi_q^2\Pi_1^{-1} & \Pi_1 + \pi_q^2\Pi_1^{-1} & 0 \\ 0 & 0 & 2\pi_q \end{bmatrix}.
$$

Then LMI (23) can be written as

$$
\begin{aligned}
L &= \begin{bmatrix} U_1^T & 0 & 0 \\ 0 & U_2^T & 0 \\ 0 & 0 & U_2^T \end{bmatrix} L_c(\mu) \begin{bmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_2 \end{bmatrix} \\
&+ \begin{bmatrix} V_1^T & 0 & 0 \\ 0 & V_2^T & 0 \\ 0 & 0 & V_2^T \end{bmatrix} L_o(\mu) \begin{bmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_2 \end{bmatrix} < 0
\end{aligned} \tag{27}
$$

where

$$
U_1 = \begin{bmatrix} I & 0 \end{bmatrix},\ U_2 = \begin{bmatrix} I & I & 0 \\ 0 & 0 & I \end{bmatrix},\ V_1 = \begin{bmatrix} 0 & \pi_q\Sigma^{-1} \end{bmatrix},
$$

$$
V_2 = \begin{bmatrix} \pi_q\Pi_1^{-1} & -\pi_q\Pi_1^{-1} & 0 \\ 0 & 0 & -I \end{bmatrix}.
$$

This completes the proof.

In theorem 2, we have derived an upper bound of the model reduction error. In order to get a less conservative model reduction error bound, it is necessary for $n-k$ smallest eigenvalues of Σ and $q-v$ smallest eigenvalues of Π to be small. Hence we choose a cost function as $J = tr(PQ) + \alpha tr(RS)$ for a positive constant α. Thus, we minimize the non-convex cost function subject to the convex constraints (9) and (11). Since this optimization problem is non-convex, the optimization problem is very difficult to solve it. So we suggest an alternative suboptimal procedure using an iterative method. We summarize an iterative method to solve a suboptimal problem.

step 1 : Set $i=0$. Initialize P_i, Q_i, R_i and S_i such that $tr(P_i+Q_i) + \alpha tr(R_i+S_i)$ is minimized subject to LMI's (9) and (11).

step 2 : Set $i = i+1$.

1) Minimize $J_i = tr(P_{i-1}Q_i) + \alpha tr(R_iS_{i-1})$ subject to LMI (9).

2) Minimize $J_i = tr(P_iQ_i) + \alpha tr(R_iS_i)$ subject to LMI (11).

step 3 : If $|J_i - J_{i-1}|$ is less than a small tolerance level, stop iteration. Otherwise, go to step 2.

V. A Numerical Example

We consider following nonlinear system :

$$
\begin{aligned}
y(t)^{(3)} &= -3y(t) - 3y(t)^3 + y(t)\sin(\dot{y}(t)) - 17\dot{y}(t) \\
&\quad - 7\ddot{y}(t) + (0.1\cos(\ddot{y}(t)) + \sin(\dot{y}(t)))\ddot{y}(t) \\
&\quad + u(t)
\end{aligned} \tag{28}
$$

Assuming that $|y(t)| \le 1$, we rewrite (28) in a T-S fuzzy system with time varying uncertain parameters.

Plant Rule i ($i=1,2$) :
IF $y(t)$ is M_{i1}
THEN

$$
\begin{aligned}
\dot{x}(t) &= A_ix(t) + F_iw(t) + B_iu(t) \\
z(t) &= H_ix(t) + J_iu(t) \\
y(t) &= C_ix(t) + G_iw(t) \\
w(t) &= \Theta(t)z(t)
\end{aligned} \tag{29}
$$

where
$M_{11} = 1 - y(t)^2$, $M_{21} = y(t)^2$, $\Theta(t) = diag(\theta_1(t), \theta_2(t))$, $\theta_1(t) = \cos(\ddot{y}(t))$, $\theta_2(t) = \sin(\dot{y}(t))$ and

$$
x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -7 & -17 \end{bmatrix},
$$

$$
A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -7 & -17 \end{bmatrix},\ B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},
$$

$$F_1 = F_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1 & 1 \end{bmatrix},\ H_1 = H_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

$$J_1 = J_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},\ G_1 = G_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

We choose $\alpha = 1$ for convenience. Using the iterative method described in section IV, we obtain suboptimal solutions P, Q, R, S. From P, Q, R, S, we obtain the balanced gramian $\Sigma = diag(0.6300, 0.65260, 0.0390)$ and $\Pi = diag(0.4608, 0.1722)$. In the balanced system, the time varying parameter matrix becomes $\Theta_b(t) = W^{-1}\Theta(t)W = diag(\theta_2(t), \theta_1(t))$. By truncating 1 state and 1 parameter in the balanced system, we obtain following reduced T-S fuzzy system (30).

Plant Rule i ($i = 1, 2$) :

IF $y(t)$ is M_{i1}

THEN

$$\begin{aligned} \dot{x}(t) &= A_{ri}x(t) + F_{ri}w(t) + B_{ri}u(t) \\ z(t) &= H_{ri}x(t) + J_{ri}u(t) \\ y(t) &= C_{ri}x(t) + G_{ri}w(t) \\ w(t) &= \Theta_r(t)z(t), \end{aligned} \tag{30}$$

where

$M_{11} = 1 - y(t)^2,\ M_{21} = y(t)^2,\ \Theta_r(t) = \theta_2(t)$, $\theta_2(t) = \sin(\dot{y}(t))$ and

$$A_{r1} = \begin{bmatrix} -0.1082 & 0.3754 \\ -0.3752 & -0.2912 \end{bmatrix}, A_{r2} = \begin{bmatrix} -0.2978 & 0.5611 \\ -0.5622 & -0.1080 \end{bmatrix},$$

$$B_{r1} = B_{r2} = \begin{bmatrix} 0.2504 \\ 0.2470 \end{bmatrix}, F_{r1} = F_{r2} = \begin{bmatrix} 0.2270 \\ 0.2239 \end{bmatrix},$$
$$H_{r1} = H_{r2} = \begin{bmatrix} 0.2343 & -0.2069 \end{bmatrix},$$
$$J_{r1} = J_{r2} = 0,\ G_{r1} = G_{r2} = 0,$$
$$C_{r1} = C_{r2} = \begin{bmatrix} 0.2523 & -0.2472 \end{bmatrix}.$$

From theorem 2, we can expect the model reduction error is bounded by 0.4224.

VI. Concluding Remark

In this paper, we have studied a balanced model reduction problem for T-S fuzzy systems with time varying parameters. For this purpose, we have defined generalized controllability and observability gramians for the uncertain fuzzy system. This generalized gramians can be obtained from solutions of LMI problem. Using the generalized gramians, we have derived a balanced state space realization. We have obtained the reduced model of the fuzzy system by truncating not only some state variables but also some uncertain parameters.

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References

[1] B. C. Moore : Principal component analysis in linear systems: Controllability, observability and model reduction. IEEE Trans. Automatic Contr., vol.26, pp.17-32, 1982

[2] L. Pernebo and L. M. Silverman : Model reduction via balanced state space representations. IEEE Trans. Automatic Contr., vol.27, pp.382-387, 1982

[3] K. Glover : All optimal Hankel-norm approximations of linear multivariable systems and their error bounds. Int. J. Control, vol.39 pp.1115-1193, 1984

[4] Y. Liu and B. D. O. Anderson : Singular perturbation approximation of balanced systems. Int. J. Control, vol.50, pp.1379-1405. 1989

[5] C. L. Beck, J. Doyle and K. Glover : Model reduction of multidimensional and uncertain systems. IEEE Trans. Automatic Contr., vol.41 pp. 1466-1477, 1996

[6] G. D. Wood, P. J. Goddard and K. Glover : Approximation of linear parameter varying systems. Proceedings of the 35th CDC, Kobe, Japan, Dec. pp.406-411, 1996

[7] F. Wu : Induced L_2 norm model reduction of polytopic uncertain linear systems. Automatica, vol.32, No.10, pp.1417-1426, 1996

[8] W.M. Haddad and V. Kapila : Robust, reduced order modeling for state space systems via parameter dependent bounding functions. Proceedings of American control conference, Seattle, Washington, June pp.4010-4014, 1996

[9] K. Tanaka, T.Ikeda, H.O.Wang : Robust stabilization of a class of uncertain nonlinear systems via fuzzy control : Quadratic stabilizability, control theory, and linear matrix inequalities. IEEE Trans. Fuzzy Systems, vol.4, no.1, Feb., pp.1-13, 1996

[10] S.K.Nguang, P.Shi : Fuzzy output feedback control design for nonlinear systems : an LMI approach. IEEE Trans. Fuzzy Systems, vol.11, no.3, June pp.331-340, 2003

[11] H.D. Tuan, P. Apkarian, T. Narikiyo and Y. Yamamoto : Parameterized linear matrix inequality techniques in fuzzy control system design. IEEE Trans. Fuzzy Systems, vol.9, no.2, April, pp.324-332, 2001