An axiomatization of the Shapley value and interaction index for games on lattices

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***Abstract*— Games on lattices generalize classical cooperative games (coalitional games), bi-cooperative games, multichoice games, etc., and provide a general framework to define actions of players in a cooperative game. We provide here an axiomatization of the Shapley value and interaction index for such games.**

I. Introduction

Cooperative game theory [3], [16] deals with set functions $v : 2^N \longrightarrow \mathbb{R}$, where N is the set of players, and $v(S)$ is the *worth* of coalition of players $S \subseteq N$. Depending on the context, the worth of S may represent the monetary value of the output produced by cooperation between its members, the cost of a project designed solely for the members of S, or the abstract power of coalition S on a voting or decision system [3].

Yet, to model the real world, more sophisticated concepts are often necessary. *Ternary voting games* allow the player (voter) to vote in favor, against, or abstain [4], and similarly *bi-cooperative games* [1], [13] allow each player to be either defenders, defeaters or not participate. More generally, each player could be allowed to have m possible actions at his/her disposal. When these actions are totally ordered, we speak of *multichoice games* [12]. In this paper, we consider that the set of possible actions of a player is a distributive lattice, which could be different for each player. Clearly, all preceding concepts of games are particular cases of this general framework. Other examples of non classical cooperative games, which do not fit our framework, are cited in [7].

We intend here to define axiomatically the Shapley value and interaction index of this general class of games, and to clarify some subclasses of such games. This work was initialized in [10], [7].

II. Prerequisites

Let $N := \{1, \dots, n\}$ be the finite set of players. A *game* on N is any function $v : 2^N \longrightarrow \mathbb{R}$, such that $v(\emptyset) = 0$. We call *coalition* any subset of N. We denote $\mathcal{G}(2^N)$ the set of all games on N.

A *value* or *solution concept* is any function $\phi : \mathcal{G}(2^N) \longrightarrow \mathbb{R}^N$, which represent an assignment of income to each player. A famous example is the Shapley value [17], defined by:

$$\phi^v(i) := \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!}[v(S \cup i) - v(S)], i \in N, v \in \mathcal{G}(2^N) \quad (1)$$

Generalizing the idea of value to several players, one comes to the notion of *interaction*, originally proposed by Owen [15] and later by Murofushi and Soneda [14] for two players, and generalized by Grabisch [6]. For any $S \subseteq N$, the interaction index between players in S is defined as:

$$I^v(S) := \sum_{T \subseteq N \setminus S} \frac{(n-t-s)!t!}{(n-s+1)!} \sum_{K \cup S} (-1)^{|S \setminus K|} v(K \cup T). \quad (2)$$

A *bi-cooperative game* is a real-valued function on $\mathcal{Q}(N) := \{(S,T) \in (2^N)^2 \mid S \cap T = \emptyset\}$ (which is isomorphic to 3^N), such that $v(\emptyset, \emptyset) = 0$. $v(S,T)$ is the worth of bi-coalition (S,T), where S is the defender part and T the defeater part. Players outside $S \cup T$ do not participate. A Shapley value has been defined for each player i, and according if i plays in the defender (denoted $\phi^v_+(i)$) or defeater part (denoted $\phi^v_-(i)$) [9]:

$$\begin{aligned}\phi^v_+(i) = \sum_{S \in N \setminus i} \frac{(n-s-1)!s!}{n!}&[v(S \cup i, N \setminus (S \cup i)) \\ &- v(S, N \setminus (S \cup i))] \quad (3)\end{aligned}$$

$$\phi^v_-(i) = \sum_{S \in N \setminus i} \frac{(n-s-1)!s!}{n!}[v(S, N \setminus (S \cup i)) - v(S, N \setminus S)]. \quad (4)$$

Based on this, an interaction index for a bi-coalition (S,T) has been defined as well.

Let $(L, \leq)$ be a finite lattice [2], we denote as usual $\top, \bot, \vee, \wedge$ the top, bottom elements, and supremum and infimum. x *covers* y (denoted $x \succ y$) if $x > y$ and there is no z such that $x > z > y$. $Q \subseteq L$ is a *downset* of L if $x \in Q$ and $y \leq x$ imply $y \in Q$. The lattice is *distributive* if $\vee, \wedge$ obey distributivity. An element $j \in L$ is *join-irreducible* if it is not the bottom element and it cannot be expressed as a supremum of other elements. Equivalently j is join-irreducible if it covers only one element. Join-irreducible elements covering $\bot$ are called *atoms*, and the lattice is *atomistic* if all join-irreducible elements are atoms. The set of all join-irreducible elements of L is denoted $\mathcal{J}(L)$.

An important property is that in a distributive lattice, any element x can be written as an irredundant supremum of join-irreducible elements in a unique way (this is called the *minimal decomposition* of x).

In a finite setting, *Boolean lattices* are of the type 2^N for some set N, i.e. they are isomorphic to the lattice of subsets of some set, ordered by inclusion. Boolean lattices are atomistic, and atoms corresponds to singletons. A *linear lattice* is such that $\leq$ is a total order. All elements are join-irreducible, except $\perp$.

Given lattices $(L_1, \leq_1), \ldots, (L_n, \leq_n)$, the product lattice $L = L_1 \times \cdots \times L_n$ is endowed with the product order $\leq$ of $\leq_1, \ldots, \leq_n$ in the usual sense. Elements of x can be written in their vector form $(x_1, \ldots, x_n)$. We use the notation (x_A, y_{-A}) to indicate a vector z such that $z_i = x_i$ if $i \in A$, and $z_i = y_i$ otherwise. Similarly L_{-i} denotes $\prod_{j \neq i} L_j$, while $L_K := \prod_{j \in K} L_j$. All join-irreducible elements of L are of the form $(\perp_1, \ldots, \perp_{j-1}, i_0, \perp_{j+1}, \ldots, \perp_n)$, for some j and some join-irreducible element i_0 of L_j. A *vertex* of L is any element whose components are either top or bottom. We denote $\Gamma(L)$ the set of vertices of L. Note that $\Gamma(L) = L$ iff L is Boolean.

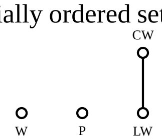
Let $(L, \leq)$ be some finite lattice, and consider $f : L \longrightarrow \mathbb{R}$, and a join-irreducible element i of L. The *derivative* of f w.r.t i at point x is defined as [8]: $\Delta_i f(x) := f(x \vee i) - f(x)$. This definition can be considered as a *first-order* derivative. One can iterate the definition, taking several join-irreducible elements. If the lattice is distributive, since any element y can be decomposed in a minimal and unique way on join-irreducible elements, the derivative w.r.t. y, denoted $\Delta_y f$, $\forall y \in L$, can be defined as well. The derivative $\Delta_i f(x)$ is said to be *Boolean* if $x \vee i \succ x$ (or more generally $\Delta_y f(x)$ is Boolean if $[x, x \vee y]$ is a Boolean lattice).

III. A GENERAL FRAMEWORK FOR GAMES ON LATTICES

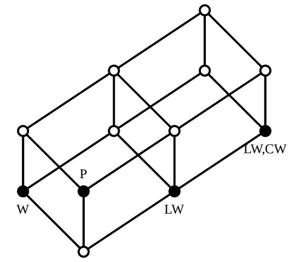
Definition 1: We consider distributive finite lattices $L_1, \ldots, L_n$ and their product $L := L_1 \times \cdots \times L_n$. A *game on* L is any function $v : L \longrightarrow \mathbb{R}$ such that $v(\perp) = 0$. The set of such games is denoted $\mathcal{G}(L)$. A game is *monotone* if $x \leq x'$ implies $v(x) \leq v(x')$.

Clearly, all previous examples of games fall into this category. We address now the question of the exact meaning of the L_i's and how they are built. Each player $i \in N$ has at his/her disposal a set of *elementary* or *pure* actions $j_1, \ldots, j_{n_i}$. These elementary actions are partially ordered (e.g. in the sense of benefit caused by the action), forming a partially ordered set $(\mathcal{J}_i, \leq)$. Then the set $(\mathcal{O}(\mathcal{J}_i), \subseteq)$ of downsets of $\mathcal{J}_i$ is a distributive lattice denoted L_i, whose join-irreducible elements correspond to the elementary actions. The *bottom action* $\perp$ of L_i is the action which amounts to do nothing. Hence, each action in L_i is either a pure action j_k or a combined action $j_k \vee j_{k'} \vee j_{k''} \vee \cdots$ consisting of doing all actions $j_k, j_{k'}, \ldots$ for player i.

For example, assume that players are gardeners who take care of some garden or park. Elementary actions are watering (W), light weeding (LW), careful weeding (CW), and pruning (P). All these actions are benefic for the garden and clearly LW<CW, but otherwise actions seem to be incomparable. They form the following partially ordered set:



which in turn form the following lattice of possible actions:



In the sequel, we consider several particular cases of games on lattices. The first one is the case where L is a *product of linear lattices* $L_1, \ldots, L_n$, with $L_i := \{0, 1, 2, \ldots, l_i\}$ (we call them *linear games*, they coincide with multichoice games). With some abuse of notation, for some $k \in L_i$ we write k_i for $(0_{-i}, k_i)$, i.e. $(0, \ldots, 0, k, 0 \ldots, 0)$, where k is at the ith position.

The second case is when $L = \prod_{i=1}^n L_i$, with L_i being *linear reflection lattices*, denoted $\{-l_i, \ldots, -1, 0, 1, \ldots, l_i\}$. We denote $\top_i, \perp_i$ top and bottom of L_i, join-irreducible elements are as usual $-l_i + 1, \ldots, l_i$. Although the L_i's here are isomorphic to (ordinary) linear lattices, we distinguish the 0 level as the level where no action is performed, and levels with negative values are considered as harmful or against the coalition. Hence, a *bipolar game* is any function $v : L \longrightarrow \mathbb{R}$ such that $v(0_N) = 0$. Bi-cooperative games in the usual sense are bipolar games with $L_1 = \cdots = L_n = \{-1, 0, 1\}$. We denote by $\mathcal{G}_{\pm}(L)$ the set of bipolar games.

IV. AXIOMATIZATION OF THE SHAPLEY VALUE

A. Case of linear games

Our aim is to define $\phi^v(k_i)$, for any $i \in N$, any $k \in L_i, k \neq 0$ (i.e. k_i ranges over all join-irreducible elements of L). For some $k \in L_i$, $k \neq 0$, player i is said to be k-*dummy* if $v(x, k_i) - v(x, (k-1)_i) = v(k_i) - v((k-1)_i)$, for any $x \in L_{-i}$.

> **Dummy axiom (D):** $\forall v \in \mathcal{G}(L)$, for all join-irreducible k_i, $\phi^v(k_i) = v(k_i) - v((k-1)_i)$ if i is k-dummy.

For some $k \in L_i$, $k \neq 0$, player i is said to be k-*null* if $v(x, k_i) = v(x, (k-1)_i)$, for any $x \in L_{-i}$.

> **Null axiom (N):** $\forall v \in \mathcal{G}(L)$, for all join-irreducible k_i, $\phi^v(k_i) = 0$ if i is k-null.

It is easy to see that the dummy axiom is stronger than the null axiom.

> **Linear axiom (L):** ϕ^v is linear on the set of games, i.e. for any join-irreducible k_i
>
> $$\phi^v(k_i) = \sum_{x \in L} a_x^{k_i} v(x)$$
>
> with $a_x^{k_i} \in \mathbb{R}$.

Proposition 1: Under **(L)** and **(N)**, $\forall v \in \mathcal{G}(L)$, for all join-irreducible k_i,

$$\phi^v(k_i) = \sum_{x \in L_{-i}} p_x^{k_i} \Delta_{k_i} v(x, (k-1)_i),$$

with $p_x^{k_i} \in \mathbb{R}$.

We recall that $\Delta_{k_i}v(x)$ is the derivative of v w.r.t. the join-irreducible element k_i, precisely $v(x\vee k_i)-v(x)$. Let us see what we get in addition with the dummy axiom.

Proposition 2: Under **(L)** and **(D)**, $\forall v\in\mathcal{G}(L)$, for all join-irreducible k_i,

$$\phi^v(k_i)=\sum_{x\in L_{-i}}p_x^{k_i}\Delta_{k_i}v(x,(k-1)_i),$$

with $p_x^{k_i}\in\mathbb{R}$, and $\sum_{x\in L_{-i}}p_x^{k_i}=1$.

Monotonicity axiom (S): if v is monotone, then $\phi^v(k_i)\geq 0$, for all join-irreducible k_i.

Proposition 3: Under axioms **(L)**, **(N)** and **(M)**, $\forall v\in\mathcal{G}(L)$, for all join-irreducible k_i,

$$\phi^v(k_i)=\sum_{x\in L_{-i}}p_x^{k_i}\Delta_{k_i}v(x,(k-1)_i),$$

with $p_x^{k_i}\geq 0$.
Let σ be a permutation on N. With some abuse of notation we write $\sigma(x):=(x_{\sigma(1)},\dots,x_{\sigma(n)})$.

Symmetry axiom (S): $\phi_{v\circ\sigma^{-1}}(\sigma(k_i))=\phi^v(k_i)$, for any game v, any join-irreducible k_i.

Proposition 4: Under **(L)**, **(N)** and **(S)**,

$$\phi^v(k_i)=\sum_{x\in L_{-i}}p^k_{n_1,\dots,n_l}\Delta_{k_i}v(x,(k-1)_i)$$

where $l:=\max(l_1,\dots,l_n)$, and n_j is the number of components of x being equal to j.

Invariance axiom (I) Let us consider v_1,v_2 on L such that for some $i\in N$,

$$v_1(x,x_i)=v_2(x,\underline{x_i}),\quad\forall x\in L_{-i},\forall x_i\in L_i,x_i\neq 0,$$

where $\underline{x_i}$ means the element covered by x_i (in case it is unique). Then for any such games, $\phi^{v_1}(k_i)=\phi^{v_2}((k-1)_i),\forall k>1$.

The axiom says that when a game (v_2) is merely a shift of another game v_1 concerning player i, the Shapley values are the same for this player. This implies that the way of computing ϕ^v does not depend on the level k, as shown in the next propositions.

Proposition 5: Under axioms **(L)** and **(I)**, $a^{k_i}_{(x,x_i)}=a^{(k-1)_i}_{(x,\underline{x_i})}$, for all $x\in L_{-i},\forall i\in N,\forall x_i,k_i\in L_i,x_i,k_i\neq 0$.

Proposition 6: Under axioms **(L)**, **(N)** and **(I)**, $p_x^{k_i}=p_x^{(k-1)_i}$, for all $x\in L_{-i},\forall i\in N,\forall k>1$.

Efficiency axiom (E): $\sum_{k_i\in\mathcal{J}(L)}\phi^v(k_i)=v(\top)$.

Proposition 7: Suppose $L_1=L_2=\cdots=L_n=:L$. Under axioms **(L)**, **(N)**, **(S)** and **(E)**, the coefficients $p^k_{n_1,\dots,n_l}$ satisfy:

$$p^l_{0,\dots,0,n-1}=1/n$$
$$n_lp^l_{n_1,\dots,n_l-1}+\sum_{j=1}^{l-1}n_j(p^j_{n_1,\dots,n_j-1,\dots,n_l}-p^{j+1}_{n_1,\dots,n_j-1,\dots,n_l})$$
$$=(n-n_1-\cdots-n_l)p^1_{n_1,\dots,n_l},$$
$$(n_1,\dots,n_l)\neq(0,\dots,0),n_l\neq n-1.$$

The final result is the following.

Theorem 1: Suppose $L_1=L_2=\cdots=L_n=:L$. Under axioms **(L)**, **(D)**, **(M)**, **(S)**, **(I)** and **(E)**,

$$\phi^v(k_i)=\sum_{x\in\Gamma(L^{n-1})}\frac{(n-n_l-1)!n_l!}{n!}[v(x,k_i)-v(x,(k-1)_i)],$$

where $\Gamma(L^{n-1})$ is the set of vertices of L^{n-1}, and n_l is the number of components of x equal to l.
Note that the above formula reduces to (3) and (4) when $L=\{-1,0,1\}^N$ and $k=1,0$ respectively.

B. Alternative view of the Shapley value

Our Shapley value obtained so far could have a strange flavor if we stick to the idea that the Shapley value $\phi^v(k_i)$ should be the reward for player i for having played at level k. If it happens that k_i is null, which means that it is the same that playing at level $k-1$ for i, it may look strange that the reward of player i is 0, while it should be equal to the reward if he had played at level $k-1$. In fact, the proposed approach gives a kind of *differential reward* w.r.t the level just below. So the usual intuition behind the Shapley value is recovered if one sums up these differential rewards from the 1st level to the current level of interest. Alternatively, one may change the axiomatization in order to get this directly. This is the aim of this section. We denote by $\Phi^v(k_i)$ this new (cumulative) Shapley value.

For some $i\in N$ and $0<k\leq l_i$, k_i is *strongly null* if $v(x,k_i)=v(x,0_i)$, for all $x\in L_{-i}$. Note that if v is monotone, then the strong nullity of k_i implies the strong nullity of all j_i, $j<k$, and the nullity of all $j_i,j\leq k$

Strong null axiom (SN): $\forall v\in\mathcal{G}(L)$, for all join-irreducible k_i, $\Phi^v(k_i)=0$ if k_i is strongly null.

Proposition 8: Under **(L)** and **(SN)**, $\forall v\in\mathcal{G}(L)$, for all join-irreducible k_i,

$$\Phi^v(k_i)=\sum_{x\in L_{-i}}p_x^{k_i}[v(x,k_i)-v(x,0_i)],$$

with $p_x^{k_i}\in\mathbb{R}$.
Join-irreducible k_i is *strongly dummy* if $v(x,k_i)=v(x,0_i)+v(k_i)$.

Strong dummy axiom (SD): $\forall v\in\mathcal{G}(L)$, for all join-irreducible k_i, $\Phi^v(k_i)=v(k_i)$ if k_i is strongly dummy.

Note that **(SD)** is stronger than **(SN)**.

Proposition 9: Under **(L)** and **(SD)**, $\forall v\in\mathcal{G}(L)$, for all join-irreducible k_i,

$$\Phi^v(k_i)=\sum_{x\in L_{-i}}p_x^{k_i}[v(x,k_i)-v(x,0_i)],$$

with $p_x^{k_i}\in\mathbb{R}$, and $\sum_{x\in L_{-i}}p_x^{k_i}=1$.
The symmetry axiom works the same as previously, so that we get:

Proposition 10: Under **(L)**, **(SN)** and **(S)**,

$$\Phi^v(k_i)=\sum_{x\in L_{-i}}p^k_{n_1,\dots,n_l}[v(x,k_i)-v(x,0_i)]$$

where $l := \max(l_1, \dots, l_n)$, and n_j is the number of components of x being equal to j.

We introduce now a variant of axiom **(I)**.

Strong invariance axiom (SI): Let us consider v_1, v_2 in $\mathcal{G}(L)$ such that for some $i \in N$:

$$v_1(x, x_i) = v_2(x, \underline{x}_i), \quad \forall x \in L_{-i}, \forall x_i > 1$$
$$v_1(x, 0_i) = v_2(x, 0_i), \quad \forall x \in L_{-i}.$$

Then $\Phi^{v_1}(k_i) = \Phi^{v_2}((k-1)_i)$, $1 < k \leq l_i$.

Proposition 11: Under **(L)**, **(SN)** and **(SI)**, we have $p_x^{k_i} = p_x^{(k-1)_i}$, for all $x \in L_{-i}$, and all $k \in L_i, 1 < k \leq l_i$.

Strong efficiency (SE): $\sum_{i \in N} \Phi^v(\top_i) = v(\top)$.

Proposition 12: Suppose $L_1 = L_2 = \cdots = L_n =: L$. Under axioms **(L)**, **(SN)**, **(S)** and **(SE)**, the coefficients $p^k_{n_1,\dots,n_l}$ satisfy:

$$p^l_{0,\dots,0,n-1} = 1/n$$
$$n_l p^l_{n_1,\dots,n_l-1} = (n - n_1 - \cdots - n_l) p^l_{n_1,\dots,n_l},$$
$$(n_1, \dots, n_l) \neq (0, \dots, 0), n_l \neq n-1.$$

The final result is the following.

Theorem 2: Suppose $L_1 = L_2 = \cdots = L_n =: L$. Under axioms **(L)**, **(SD)**, **(M)**, **(S)**, **(SI)** and **(SE)**,

$$\Phi^v(k_i) = \sum_{x \in \Gamma(L^{n-1})} \frac{(n - n_l - 1)! n_l!}{n!} [v(x, k_i) - v(x, 0_i)],$$

where $\Gamma(L^{n-1})$ is the set of vertices of L^{n-1}.

C. Case of bipolar games

The general philosophy is the following. The underlying L for bipolar games is isomorphic to the L used for games, hence up to some very minor modifications, the same axioms and recursion formula can be used for bipolar games, leading to the same expression for interaction (for bi-cooperative games, same axioms were taken, which effectively lead to the same formulas). However, the idea of bipolarity is to distinguish what is done for positive and negative parts. Hence our axioms and construction will reflect this, and lead to a different result.

Axioms **(L)**, **(D)**, **(N)**, **(A)** are not affected by the bipolar nature of the game. In contrast, the invariance and efficiency axioms have to be changed.

Bipolar Invariance axiom (BI) Let us consider v_1, v_2 on L such that for some $i \in N$,

$$v_1(x, x_i) = v_2(x, \underline{x_i}), \forall x \in L_{-i}, \forall x_i \in [-l_i+1, 0] \cup [1, l_i].$$

Then for any such games $\phi_{v_1}(k_i) = \phi_{v_2}((k-1)_i)$, $\forall k \in [-l_i+2, 0] \cup [2, l_i]$.

The axiom says that when a game (v_2) is merely a shift of another game v_1 concerning player i and positive (or negative) level k, the Shapley values are the same. This implies that the way of computing ϕ^v does not depend on the level k, but may differ on the positive and negative parts, as shown in the next proposition.

Proposition 13: Under axioms **(L)**, **(N)** and **(BI)**, $p_x^{k_i} = p_x^{(k-1)_i}$, for all $x \in L_{-i}$, $\forall i \in N$, $\forall k \in [-l_i+2, 0] \cup [2, l_i]$.

The above proposition shows that we have two sets of coefficients p^+_x and p^-_x.

Bipolar Efficiency axiom (BE): for any bipolar game v on L,

(i) $\sum_{k_i \in \mathcal{J}(L), k_i > 0} \phi^v(k_i) = v(\top) - v(0)$

(ii) $\sum_{k_i \in \mathcal{J}(L), k_i \leq 0} \phi^v(k_i) = v(0) - v(\bot)$

Note that **(BE)** implies **(E)**, but not the converse.

Proposition 14: Suppose $L_1 = L_2 = \cdots = L_n =: L$. Under axioms **(L)**, **(N)**, **(S)** and **(BE)**, the coefficients $p^k_{n-l+1,\dots,n_l}$ satisfy:

$$p^l_{0,\dots,0,n-1} = 1/n$$
$$p^{-l+1}_{0,\dots,0} = 1/n$$

$$n_l p^l_{n-l+1,\dots,n_l-1} + \sum_{j=1}^{l-1} n_j (p^j_{n-l+1,\dots,n_j-1,\dots,n_l} - p^{j+1}_{n-l+1,\dots,n_j-1,\dots,n_l}) = n_0 p^1_{n-l+1,\dots,n_0-1,\dots,n_l}, \quad n_0, n_l \neq n-1$$

$$n_0 p^0_{n-l+1,\dots,n_0-1,\dots,n_l} + \sum_{j=-l+1}^{0} n_j (p^j_{n-l+1,\dots,n_j-1,\dots,n_l} - p^{j+1}_{n-l+1,\dots,n_j-1,\dots,n_l}) = (n - n_{-l+1} - \cdots - n_l) p^{-l+1}_{n-l+1,\dots,n_l},$$
$$(n_{-l+1}, \dots, n_l) \neq (0, \dots, 0), n_0 \neq n-1.$$

Theorem 3: Suppose $L_1 = L_2 = \cdots = L_n =: L$. Under axioms **(L)**, **(D)**, **(S)**, **(M)** **(BE)**, and **(BI)**,

$$\phi^v(k_i) = \sum_{x \in L_{-i} | x_k = 0 \text{ or } \top_k} \frac{(n - n_l - 1)! n_l!}{n!} \Delta_{k_i} v(x, (k-1)_i), \quad \forall k > 0$$

$$\phi^v(k_i) = \sum_{x \in L_{-i} | x_k = \bot_k \text{ or } 0} \frac{(n - n_0 - 1)! n_0!}{n!} \Delta_{k_i} v(x, (k-1)_i), \quad \forall k \leq 0.$$

Note that this does not correspond to the Shapley values defined for bi-cooperative games. In fact, what we get is a complete separation between the positive and negative parts since there is no element in the summations mixing positive and negative components.

Since linear reflection lattices are isomorphic to linear lattices, axiom **(E)** can be used as well and Prop. 7 applies. However, axiom **(BI)** entails the existence of two sets of coefficients p^+_x, p^-_x, so that adding axioms **(D)** and **(M)** there is no more a unique solution for the coefficients. Hence axiom **(BE)** is (mathematically) suited to **(BI)**.

D. General case

We consider some x_i, join-irreducible in L_i, and we try to obtain $\phi^v(x_i)$.

Linear axiom (L): ϕ^v is linear on the set of games, i.e. for any join-irreducible $x_i \in L_i$, $\phi^v(x_i) = \sum_{y\in L} a_y^{x_i} v(y)$ with $a_y^{x_i} \in \mathbb{R}$.

We say that x_i is *null* for v if $v(y, x_i) = v(y, \underline{x_i})$ for every $y \in L_{-i}$.

Null axiom (N): If x_i is null for v, then $\phi^v(x_i) = 0$.

Proposition 15: Under **(L)** and **(N)**, $\forall v \in \mathcal{G}(L)$, for all join-irreducible $x_i \in L_i$,

$$\phi^v(x_i) = \sum_{y\in L_{-i}} p_y^{x_i} \Delta_{x_i} v(y, \underline{x_i}).$$

V. Axiomatization of interaction index

The approach presented here is based on recursion formulae, starting from the Shapley value, as in [11]. Other approaches, like the one of Fujimoto [5] based on a partnership axiom, are possible, and will be presented in forthcoming papers. Our presentation will restrict to the case where all L_i's are linear. Our aim is to derive an expression of the interaction index $I^v(x)$, for any $x \in L$.

A. Basic expression with derivatives

We will show in this section that the interaction index takes naturally the form of an average derivative.

Linear axiom (L): I^v is linear on the set of games, i.e. for any $x \in L$

$$I^v(x) = \sum_{y\in L} a_y^x v(y)$$

with $a_y^x \in \mathbb{R}$.

Null axiom (N): Assume k_i is null for v. Then for any $x \geq k_i$, $I^v(x) = 0$.

Note that this is a generalization of the null axiom for the Shapley value.

Dummy axiom (D): Assume k_i is dummy for v. Then for any $x \geq k_i$, $I^v(x) = 0$.

Again, the dummy axiom implies the null axiom.

Proposition 16: Under **(L)** and **(N)**, $\forall v \in \mathcal{G}(L)$, for all $x \in L$,

$$I^v(x) = \sum_{y \in L_{N\setminus T}} p_y^x \Delta_x v(y, \underline{x}),$$

where $T := \{i \in N \mid x_i \neq 0\}$.

This result encompasses the result for classical games, which was proven with the dummy axiom [11]. In fact the null axiom suffices.

B. Recursion formula for the linear case

By analogy with [11], we propose a recursion formula, which computes $I^v(x)$ for any $x \in L$ from the expression of $I^v(k_i)$, for any join-irreducible k_i in L. We need some definitions (see [10] for proofs and details).

For any $x \in L$, we denote $J := \{k \in N \mid x_k \neq \perp_k\}$. For a given $x \in L$ and its associated $J \subseteq N$, we introduce new concepts of games. For any $K \subseteq J, K \neq \emptyset, J$, the function v *restricted to* $\prod_{k\in N\setminus K} L_k$ is denoted $v_x^{N\setminus K}$ and defined by:

$$v_x^{N\setminus K}(y) := v(y'), \text{ with } y'_k := \begin{cases} \underline{x_k}, & \text{if } k \in K \\ y_k, & \text{else} \end{cases}, \forall y \in \prod_{k\in N\setminus K} L_k.$$

The function v *reduced to* x is a function $v^{[x]}$ defined on $\prod_{k\in N\setminus J} L_k \times \{\perp_{[x]}, \top_{[x]}\}$ by:

$$v^{[x]}(y) := v(\psi_{[x]}(y)), \quad \forall y \in \prod_{k\in N\setminus J} L_k \times \{\perp_{[x]}, \top_{[x]}\},$$

and $\psi_{[x]} : \prod_{k\in N\setminus J} L_k \times \{\perp_{[x]}, \top_{[x]}\} \longrightarrow L$ is defined by

$$\psi_{[x]}(y) := y', \text{ with } y'_k := \begin{cases} x_k, & \text{if } k \in J \text{ and } y_{[x]} = \top_{[x]} \\ \underline{x_k}, & \text{if } k \in J \text{ and } y_{[x]} = \perp_{[x]} \\ y_k, & \text{if } k \notin J. \end{cases}$$

We propose the following recursion formula:

$$I^v(x) = I^{v^{[x]}}(\perp_{N\setminus J}, \top_{[x]}) - \sum_{K\subset J, K\neq\emptyset} I^{v_x^{N\setminus K}}(x_{|N\setminus K}), \quad (5)$$

where $\perp_{N\setminus J}$ stands for the vector $(\perp_k)_{k\in N\setminus J}$, and $x_{|N\setminus K}$ is the restriction of x to coordinates in $N \setminus K$. In [10], it was supposed that the interaction takes the following form:

$$I^v(x) := \sum_{y \mid y_k = \top_k \text{ or } \perp_k \text{ if } k\notin J, y_k = \underline{x_k} \text{ else}} \alpha_{h(y)}^{|J|} \Delta_x v(y) \quad (6)$$

where J is defined as above, and $h(y)$ is the number of components of y equal to $\top_k$, $k = 1, \ldots, n$. Then, the following result was proved.

Theorem 4: Denoting $\alpha_k^j(n)$ the coefficients α_k^j involved into (6), the recursion formula (5) induces the following recursive relation:

$$\alpha_k^j(n) = \alpha_k^1(n-j+1), \quad \forall k = 0, \ldots, n-j, \quad \forall j = 1 \ldots, n. \quad (7)$$

Note that $\alpha_k^j(n)$ depends only on k and $n-j$.

Using (7), and imposing that $\alpha_k^1(n) = \frac{(n-k-1)!k!}{n!}$, we get

$$\alpha_k^j = \frac{(n-j-k)!k!}{(n-j+1)!}.$$

which coincide with the coefficients in (2).

C. The bipolar case

We can propose similar recursion formulas for the bipolar case, which generalize the ones proposed for bi-capacities [9]. Moreover, we will provide a very general result abour recursion formulas.

For any $x \in L$, we denote by $J := \{k \in N \mid x_k \neq \perp_k\}$, and J^+, J^- the subsets of J containing the positive and negative coordinates of x, namely $J^+ := \{k \in N \mid x_k > 0\}$ and $J^- := \{k \in N \mid \perp_k < x_k \leq 0\}$. We write x^+, x^- the positive and negative parts of x.

The *game v reduced to x^+* is a function $v^{[x^+]}$ defined on $\prod_{k\in N\setminus J^+} L_k \times \{0_{[x^+]}, \top_{[x^+]}\}$ by:

$$v^{[x^+]}(y) := v(\psi_{[x^+]}(y)),\ \forall y \in \prod_{k\in N\setminus J^+} L_k \times \{0_{[x^+]}, \top_{[x^+]}\},$$

and $\psi_{[x^+]} : \prod_{k\in N\setminus J^+} L_k \times \{0_{[x^+]}, \top_{[x^+]}\} \longrightarrow L$ is defined by

$$\psi_{[x^+]}(y) := y',\ \text{with } y'_k := \begin{cases} x_k, & \text{if } k \in J^+ \text{ and } y_{[x^+]} = \top_{[x^+]} \\ \underline{x_k}, & \text{if } k \in J^+ \text{ and } y_{[x^+]} = 0_{[x^+]} \\ y_k, & \text{if } k \notin J^+. \end{cases}$$

Similarly, the *game v reduced to x^-* is a function $v^{[x^-]}$ defined on $\prod_{k\in N\setminus J^-} L_k \times \{\perp_{[x^-]}, 0_{[x^-]}\}$ by:

$$v^{[x^-]}(y) := v(\psi_{[x^-]}(y)),\ \forall y \in \prod_{k\in N\setminus J^-} L_k \times \{\perp_{[x^-]}, 0_{[x^-]}\},$$

and $\psi_{[x^-]} : \prod_{k\in N\setminus J^-} L_k \times \{\perp_{[x^-]}, 0_{[x^-]}\} \longrightarrow L$ is defined by

$$\psi_{[x^-]}(y) := y',\ \text{with } y'_k := \begin{cases} x_k, & \text{if } k \in J^- \text{ and } y_{[x^-]} = 0_{[x^-]} \\ \underline{x_k}, & \text{if } k \in J^- \text{ and } y_{[x^-]} = \perp_{[x^-]} \\ y_k, & \text{if } k \notin J^-. \end{cases}$$

We propose two recursion formulas:

$$I^v(x) = I^{v^{[x^+]}}(\perp_{N\setminus J}, x_{J^-}, \top_{[x^+]}) - \sum_{K\subset J^+, K\neq\emptyset} I^{v_x^{N\setminus K}}(x_{|N\setminus K}) \tag{8}$$

$$I^v(x) = I^{v^{[x^-]}}(\perp_{N\setminus J}, 0_{[x^-]}, x_{J^+}) - \sum_{K\subset J^-, K\neq\emptyset} I^{v_x^{N\setminus K}}(x_{|N\setminus K}) \tag{9}$$

where $\perp_{N\setminus J}$ stands for the vector $(\perp_k)_{k\in N\setminus J}$, (similarly for $\top, 0$), and $x_{|N\setminus K}$ is the restriction of x to coordinates in $N \setminus K$. The following is the main result of this section and encompasses all previous results on recursive formulas.

Theorem 5: Let $\{L^+(K)\}_{K\subset N, K\neq\emptyset}$, $\{L^-(K)\}_{K\subset N, K\neq\emptyset}$, and $\{L^\pm(K)\}_{K\subset N, K\neq\emptyset}$ be arbitrary families of non empty subsets of L_K, $K \subset N, K \neq \emptyset$. Assume recursive formulas (8) and (9) hold, and

$$I^v(k_i) = \sum_{x\in L^+(N\setminus i)} \alpha_x^{1,0}(n)\Delta_{k_i} v(x, \underline{k_i}), \quad k_i > 0$$

$$I^v(k_i) = \sum_{x\in L^-(N\setminus i)} \alpha_x^{0,1}(n)\Delta_{k_i} v(x, \underline{k_i}), \quad k_i \leq 0$$

$$I^v(k_i, k_j, \perp_{N\setminus ij}) = \sum_{x\in L^\pm(N\setminus\{i,j\})} \alpha_x^{1,1}(n)\Delta_{k_i\vee k_j} v(x, \underline{k_i}, \underline{k_j}),$$
$$k_i > 0, k_j \leq 0$$

Then

$$I^v(x_{J^+}, \perp_{N\setminus J^+}) = \sum_{x\in L^+(N\setminus J^+)} \alpha_x^{|J^+|}(n)\Delta_{(x_{J^+}, \perp_{N\setminus J^+})} v(x, \underline{x_{J^+}})$$

$$I^v(x_{J^-}, \perp_{N\setminus J^-}) = \sum_{x\in L^-(N\setminus J^-)} \beta_x^{|J^-|}(n)\Delta_{(x_{J^-}, \perp_{N\setminus J^-})} v(x, \underline{x_{J^-}})$$

$$I^v(x_{J^+}, x_{J^-}, \perp_{N\setminus(J^+\cup J^-)}) = \sum_{x\in L^\pm(N\setminus(J^+\cup J^-))} \gamma_x^{|J^+|+|J^-|}(n) \times \Delta_{(x_{J^+}, x_{J^-}, \perp_{N\setminus(J^+\cup J^-)})} v(x, \underline{x_{J^+}}, \underline{x_{J^-}})$$

and

$$\alpha_x^1 = \alpha_x^{1,0}, \quad \alpha_x^j(n) = \alpha_x^{j-1}(n-1), \quad \forall x, 1 \leq j \leq n, \forall n \geq 1, \tag{10}$$

and similarly for β_x^j, and

$$\gamma_x^2 = \alpha_x^{1,1}, \quad \gamma_x^j(n) = \gamma_x^{j-1}(n-1), \quad \forall x, 2 \leq j \leq n, \forall n \geq 2. \tag{11}$$

The proof is based on the following general result for Boolean derivatives.

Lemma 1: Let $(L, \leq)$ be a finite lower locally distributive lattice, and $f : L \longrightarrow \mathbb{R}$ a real-valued function on it. Assume $x, y \in L$ are such that the derivative $\Delta_y v(x)$ is Boolean. The following holds.

(i) $\Delta_y f(x) = \sum_{z\in[x,x\vee y]} (-1)^{h(x\vee y)-h(z)} f(z)$

(ii) $\Delta_y f(x) = f(x \vee y) - f(x) - \sum_{\substack{y'<y \\ y'\neq\perp}} \Delta_{y'} f(x)$,

where h is the height function of L.

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