Partially Symmetric Bi-Capacities in MCDM

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***Abstract*— Recently, the concept of bi-capacity has been introduced in MCDM as a generalization on bipolar scales of capacities. This new model has the ability to represent behavioral changes between attractive and repulsive values w.r.t. criteria. The main drawback of this model is that it holds a huge number of parameters, which makes its determination quite delicate in practice. From the observation that bipolarity is usually required only on a subset of criteria, we define the notion of P-symmetric bi-capacity that has the complexity of a bi-capacity on P and the complexity of a capacity on the remaining criteria. This concept is applied to a MCDM example.**

I. Introduction

MultiCriteria Decision Making (MCDM) aims at modeling the preferences $\succeq$ of a decision maker (DM) over alternatives described by several points of view $X_1, \dots, X_n$. Two fundamental operations are needed in approaches to MCDM problems: the construction of utility functions and the aggregation of these utility functions.

Utility functions depict the preferences $\succeq_i$ of the DM over one attribute X_i. It may exist on each attribute a particular element, called *neutral level*, such that better elements are considered as *good* whereas worst elements are considered as *bad* for the DM. Such a neutral level exists whenever relation $\succeq_i$ corresponds to two opposite notions of common language. For example, this is the case when $\succeq_i$ means “more attractive than”, “better than”, etc., whose opposite notions are respectively “attractiveness/repulsiveness”, and “good/bad”. In such cases, the measurement scale associated to the utility function is said to be of *bipolar* nature. By contrast, relation “more satisfactory than” do not clearly exhibit a neutral level. Such scale is said to be of *unipolar* nature.

In practice, most of the time, the underlying scale is unipolar and is represented by the interval $[0,1]$. However, bipolar scales are of great interest, especially for the modeling of *affect* expressed by the DM [6]. The motivation for doing so is that human DMs do effectively distinguish positive and negative assets, and behave differently.

On the other hand, most of the well-known methods (ELECTRE, MACBETH, MAUT, ...) are based on a weighted sum. This function requires the independence of the criteria, which is very rarely satisfied. When a DM evaluates an alternative, he generally makes an overall assessment by considering all criteria at the same time instead of considering them separately one at a time. The satisfaction the DM allots to the value w.r.t. one criterion depends on the values w.r.t. the other criteria. A typical example of that is the presence of intolerant (resp. tolerant) behaviour of the DM w.r.t. several criteria for which all of them must be well-satisified (resp. only one of them shall be well-satisfied) in order to get a fine overall assessment. To model such behaviors, the Choquet integral has been introduced in MCDM as an aggregation operator. It has indeed been shown to represent a certain kind of interaction between criteria, ranging from redundancy (tolerance - negative interaction) to synergy (intolerance - positive interaction).

For bipolar scales, extensions of the Choquet integral has been introduced: the asymmetric Choquet integral, and more generally the CPT (Cumulative Prospect Theory) in decision under risk or uncertainty [7]. Despite the ability of these models to cope with many decision behaviors, it is not uncommon to meet practical situations where these models fail to represent preferences, even though these preferences seem rather natural. Section 4 presents one of such an example. These models fail to represent situations where the DM has different decision strategies on attractive and repulsive values. These are examples of interaction between criteria on bipolar scales.

An extension of the Choquet integral and the CPT model for bipolar scales has recently been proposed [2]. As recalled in Section 4, this new model is able to represent sign-dependant decision behaviors. It leads to a generalization of the notion of capacity called bi-capacity [3].

The main asset of bi-capacities is to take into account explicitly the sign of the criteria values. However, this versatility and flexibility has a cost: a bi-capacity holds much more parameters than a capacity. A bi-capacity contains indeed 3^n unknowns instead of 2^n for a capacity, which makes the determination of a bi-capacity quite delicate. As an example, with 5 criteria, a capacity has $2^5 = 32$ coefficients whereas a bi-capacity holds $3^5 = 243$ coefficients. Ten well-chosen learning examples are generally enough to determine a capacity with 5 criteria. It would require maybe 80 learning examples to determine a bi-capacity with 5 criteria. This is obviously beyond what a human being could stand.

To solve this problem, the idea is to consider a model that has fewer coefficients than bi-capacities. We first notice that in most MCDM problems with sign-dependant decision strategies, the bipolar nature is not generally compulsory on all criteria. In the example given in Section 4, a bipolar model is needed only for one criterion. A usual unipolar model (based on a capacity) is enough on the other criteria. Our approach consists in allowing more degrees of freedom on some criteria compared to the other ones. This is done by enforcing some symmetry properties on the criteria that do not need bipolarity. These symmetry conditions state that the interaction between positive and negative values vanishes for the criteria that do not need bipolarity. This is derived from a property satisfied by the asymmetric Choquet integral and the CPT model. We obtain a family of bi-capacities ranging from the CPT model to general bi-capacities.

II. Preliminaries

The set of all criteria is denoted by $N = \{1, \ldots, n\}$. The problem of the determination of utility functions through a bi-capacity has already been addressed in [2]. We focus here on the aggregation model so that all scores w.r.t. criteria are supposed to be given in the bipolar scale $\mathbb{R}^n$. Hence alternatives are elements of $X = \mathbb{R}^n$. Considering two acts $x, y \in X$ and $A \subset N$, we use the notation (x_A, y_{-A}) to denote the compound act $z \in X$ such that $z_i = x_i$ if $i \in A$ and y_i otherwise.

A. Capacity and Choquet integral

A *capacity*, also called fuzzy measure, is a set function $\nu : 2^N \to \mathbb{R}$ satisfying

- $A \subset B \Rightarrow \nu(A) \leq \nu(B)$,
- $\nu(\emptyset) = 0$, $\nu(N) = 1$.

In MCDM, $\nu(A)$ is interpreted as the overall assessment of the binary act $(1_A, 0_{-A})$. The set of all capacities defined on N is denoted by $\mathcal{G}^2$.

The Choquet integral defined w.r.t. a capacity ν has the following expression :

$$C_\nu(x_1, \ldots, x_n) = x_{\pi(1)}\nu(N) + \sum_{i=2}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) \nu\left(A_{\pi(i)}\right) , \tag{1}$$

where $x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(n)}$, $A_{\pi(i)} = \{\pi(i), \cdots, \pi(n)\}$ and $x_1, \ldots, x_n \in \mathbb{R}_+$.

The Choquet has been introduced in MCDM for its ability to model decision behaviors ranging from tolerance to intolerance. It has been shown to represent the importance of criteria, the interaction between criteria and other decision strategies such as veto or favor.

B. Aggregation on bipolar scales

The Choquet integral has a natural extension to bipolar scales. It has the same properties and the same mathematical expression. The Choquet integral is asymmetric in $\mathbb{R}^n$. With this model, the behavior of the DM on the whole domain is entirely determined by his behavior in the positive part, which is described by the capacity.

The Choquet model fails to represent behavioral changes between attractive and repulsive values. In decision under risk or uncertainty, the behavior of individuals is fairly different for gains (attractive values) and losses (repulsive values). It is characterized by a non-symmetric behavior. The idea is to deal the positive and the negative parts with two separate aggregation functions:

$$CPT(x) := C_{\nu_1}(x^+) - C_{\nu_2}(x^-) ,$$

where ν_1 and ν_2 are two capacities associated to the positive and the negative parts respectively. More precisely, $\nu_1(A)$ is interpreted as the overall assessment of the binary act $(1_A, 0_{-A})$, and $\nu_2(A)$ is interpreted as the opposite of the overall assessment of the binary act $(-1_A, 0_{-A})$. This is known as the Cumulative Prospect Theory (CPT) model [7]. The set of all CPT models defined on N is denoted by $\mathcal{G}^2_\pm$. In MCDM, this model amounts to weighing up the pros (attractive values) and the cons (repulsive values). There is no interaction between the positive and the negative parts. Generalizations of the CPT model are sought.

All previous models are limited by the fact that they are constructed on the notion of capacity. The idea is thus to generalize the notion of capacity. Let

$$\mathcal{Q}(N) = \{(A, B) \in \mathcal{P}(N) \times \mathcal{P}(N) \mid A \cap B = \emptyset\} .$$

A bi-capacity is a function satisfying [2]

- $A \subset A' \Rightarrow \mu(A, B) \leq \mu(A', B)$,
- $B \subset B' \Rightarrow \mu(A, B) \geq \mu(A, B')$,
- $\mu(\emptyset, \emptyset) = 0$, $\mu(N, \emptyset) = 1$, $\mu(\emptyset, N) = -1$

The first two properties depict increasingness. In MCDM, $\mu(A, B)$ is interpreted as the overall assessment of the ternary act $(1_A, -1_B, 0_{-A\cup B})$. The set of all bi-capacities on N is denoted by $\mathcal{G}^3$.

The Choquet integral w.r.t. a bi-capacity m has been proposed in [2]. Let $x \in X$, $N^+ = \{i \in N, x_i \geq 0\}$ and $B^- = N \setminus N$. Define the capacity ν by

$$\forall A \subset N\ ,\ \nu(A) := \mu\left(A \cap N^+, A \cap N^-\right) .$$

Then the Choquet integral w.r.t. μ is defined by:

$$BC_\mu(x) := C_\nu\left(x_{N^+}, -x_{N^-}\right) .$$

Let π be a permutation such that $|x_{\tau(1)}| \leq \ldots \leq |x_{\tau(n)}|$, and

$$A_i^{\pi,+} = \{\tau(i), \cdots, \tau(n)\} \cap N^+ = \left\{\tau(j)\ ,\ \tau(j) \geq \tau(i)\ , x_{\tau(j)} \geq 0\right\} ,$$

$$A_i^{\pi,-} = \{\tau(i), \cdots, \tau(n)\} \cap N^- = \left\{\tau(j)\ ,\ \tau(j) \geq \tau(i)\ , x_{\tau(j)} < 0\right\} .$$

Then one can write

$$BC_\mu(x) = \sum_{i=1}^{n} \left|x_{\tau(i)}\right| \left[\mu(A_i^{\pi,+}, A_i^{\pi,-}) - \mu(A_{i+1}^{\pi,+}, A_{i+1}^{\pi,-})\right] \tag{2}$$

Let $\Sigma_A := \{x \in \mathbb{R}^n\ , x_A \geq 0\ , x_{-A} < 0\}$. The Choquet integral w.r.t. a bi-capacity is clearly a weighted sum in each domain

$$\Sigma_A^\pi = \left\{x \in \Sigma_A\ , |x_{\tau(1)}| \leq \ldots \leq |x_{\tau(n)}|\right\} .$$

for a permutation π and $A \subset N$.

The following lemma shows that capacities correspond to asymmetric bi-capacities [2]:

Lemma 1: If m satisfies $\mu(A,B) = \mu(N \setminus B, N \setminus A)$ for all $(A,B) \in \mathcal{Q}(N)$, then the Choquet integral w.r.t. μ is equal to the Choquet integral w.r.t. the capacity ν given by $\nu(A) = \mu(A,\emptyset)$.

The CPT model associated to the capacities ν_1 and ν_2 is a particular case of bi-capacity defined by

$$\mu(A,B) = \nu_1(A) - \nu_2(B)\ . \tag{3}$$

Moreover, one has the following characterization [2]:

Lemma 2: If μ satisfies $\mu(A,B) - \mu(A,B') = \mu(A',B) - \mu(A',B')$ for all $A,A',B,B' \subset N$, such that $(A,B),(A,B'),(A',B),(A',B') \in \mathcal{Q}(N)$, then the Choquet integral w.r.t. μ reduces to a CPT model.

III. Bi-capacities with vanishing interaction indices

Bi-capacities hold much more parameters than usual capacities. Switching from a usual capacity to a bi-capacity implies indeed a tremendous increase in the number of unknowns, going from 2^n to 3^n. The idea is thus to consider intermediate models with a number of unknowns that is reasonably larger than 2^n.

Instead of bearing all the new degrees of freedom uniformly on all criteria, the idea is to put new coefficients only on the criteria that exhibit a switch of behavior between attractive and repulsive values. On the other criteria, the complexity should remain the same as for models based on capacities.

A. Towards the wished symmetry property

As said earlier, the flexibility of a bi-capacity is usually not necessary on all criteria. We wish to define bi-capacities that hold as many parameters as a usual bi-capacity on some criteria, and the same number of parameters as capacities on the remaining criteria. Let us denote by P the set of criteria for which the DM's behavior is clearly of bipolar nature.

All criteria are treated uniformly in capacities and bi-capacities. Nothing special is done in the representation model on one criterion compared to another one. The underlying lattices are indeed uniform - the 2^N uniform lattice for capacities composed of two reference levels on each criterion, and the 3^N uniform lattice for bi-capacities composed of three reference levels on each criterion. The idea is to adopt a non-uniform representation based on a non-uniform lattice, composed for instance of three levels on criteria of P and only two levels on the other criteria. The complexity of such model would be of order 3^p (with $p = |P|$) on P and 2^{n-p} (with $n = |N|$) on the other criteria, which gives an overall complexity of order $3^p \times 2^{n-p}$.

¿From Lemma 1, a capacity is a particular case of a bi-capacity that is asymmetric. In order to reduce the complexity of a bi-capacity on criteria $N \setminus P$, it seems thus natural to impose some (a)symmetric conditions on $N \setminus P$.

It is easy to characterize bi-capacities that fulfill symmetric properties on all criteria. For instance, a bi-capacity corresponds to a usual capacity if and only if

$$\forall (A,B) \in \mathcal{Q}(N) \quad \mu(A,B) = \frac{\mu(A, N \setminus A) + \mu(N \setminus B, B)}{2} \tag{4}$$

This relation concerns all elements of $\mathcal{Q}(N)$. In order to obtain symmetry only on $N \setminus P$, a natural idea is to enforce previous relation only on $\mathcal{Q}(N \setminus P)$. This gives

$$\forall (A,B) \in \mathcal{Q}(N \backslash P) \ \ \mu(A,B) = \frac{\mu(A, N \setminus A) + \mu(N \setminus B, B)}{2} \tag{5}$$

We obtain a family of bi-capacities with a gradual symmetry by choosing P from the empty set to the full set. The general bi-capacities are indeed recovered when $P = N$, and usual capacities are obtained when $P = \emptyset$. However, the resulting bi-capacity for the intermediate values of P is not so natural. This can be easily seen by investigating what we gain by putting previous relation on $\mathcal{Q}(N \backslash P)$. Relation (5) implies that the restriction of μ on $\mathcal{Q}(N \setminus P)$ is entirely determined from a knowledge of μ only on $\{(A, N \setminus A), A \subset N \setminus P\} \cup \{(N \setminus A, A), A \subset N \setminus P\}$. This set is included in $\mathcal{Q}(N) \setminus \mathcal{Q}(N \setminus P)$. Hence, the terms of μ on $\mathcal{Q}(N \setminus P)$ can be discarded, and the terms of μ on $\mathcal{Q}(N) \setminus \mathcal{Q}(N \setminus P)$ must be kept. The number of variables in the restricted model is thus $3^n - 3^{n-p}$. We see that the gain on the number of variables is far from what we are looking for.

Other attempts starting from relation (4) are also unsuccessful either due to the same reason or since the resulting model has a weird behavior.

To overcome this difficulty, another property satisfied by a symmetric bi-capacities is sought. Lemmas 1 and 2 give examples of symmetric (or asymmetric) behaviors between the positive and negative values. By (3), the decisional behavior of a CPT-model in a domain Σ_A can be deduced from that in the two domains Σ_N and $\Sigma_\emptyset$. For asymmetric bi-capacities, the behavior of the DM is determined only by his behavior on attractive values. This close link will be generally speaking lost for a general bi-capacity.

Symmetry between positive and negative values implies that there is a link between well-satisfied and ill-satisfied criteria belonging to $N \setminus P$. This can be quantified with the help of an index that measures the way the criteria interact each other within a bi-capacity. This is the interaction index.

The interaction indices have been introduced in MCDM to measure the way criteria interact each other. Let us start with the case of a capacity. Two criteria are said to interact *conjunctively* if both criteria must necessarily be satisfied together to get a good overall evaluation. The contribution of this phenomenon to the overall evaluation equals the smallest score between the two criteria. Such interaction is said to be positive. Two criteria are said to interact independently if the contribution of one criterion to the overall evaluation does not depend on the other one. The contribution of this phenomenon to the overall evaluation corresponds to a weighted sum of the scores of the two criteria. Two criteria are said to interact *substituatively* if it is enough to satisfy one criterion to get a good overall evaluation.

The contribution of this phenomenon to the overall evaluation equals the largest score between the two criteria. Such interaction is said to be negative.

When there are only two criteria, the value of this interaction (positive, zero or negative) for capacities has been defined as follows [5]:

$$I_\nu(\{i,j\}) = \nu(\{1,2\}) - \nu(\{1\}) - \nu(\{2\}) + \nu(\emptyset)$$

A conjunctive situation is indeed typically represented by the following capacity: $\nu(\{1,2\}) = 1$ and $\nu(\{1\}) = \nu(\{2\}) = \nu(\emptyset) = 0$. Previous formula yields a positive interaction. Moreover, a substituative situation is typically: $\nu(\{1,2\}) = \nu(\{1\}) = \nu(\{2\}) = 1$ and $\nu(\emptyset) = 0$. Previous formula yields then a negative interaction. Finally, an independent situation is for instance: $\nu(\{1,2\}) = 1$, $\nu(\{1\}) = \nu(\{2\}) = 1/2$ and $\nu(\emptyset) = 0$. Previous formula yields then a zero interaction. When more criteria are involved, the interaction index between criteria i and j becomes [5]:

$$I_\nu(\{i,j\}) = \sum_{K\subset N\setminus\{i,j\}} \frac{(n-k-2)!k!}{(n-1)!} \Delta_{\{i,j\}}\nu(K) \ ,$$

where $\Delta_{\{i,j\}}\nu(K) = \nu(A\cup\{i,j\}) - \nu(A\cup\{i\}) - \nu(A\cup\{j\}) + \nu(A)$. Interaction involving more than two criteria can also be defined [1]:

$$I_\nu(A) = \sum_{K\subset N\setminus A} \frac{(n-k-|A|)!k!}{(n-|A|+1)!} \Delta_A\nu(K) \ ,$$

where $\Delta_A\mu(K) = \sum_{L\subset A}(-1)^{|A|-|L|}\nu(K\cup L)$.

The notion of interaction can be generalized to bi-capacities. Let us consider once more two criteria. Then, it is natural to define four interaction indices for the parts which lead respectively to [4]:

$$\begin{aligned}
I_\nu(\{i,j\},\emptyset) &= \mu(\{1,2\},\emptyset) - \mu(\{1\},\emptyset) - \mu(\{2\},\emptyset) + \mu(\emptyset,\emptyset)\\
I_\nu(\emptyset,\{i,j\}) &= \mu(\emptyset,\emptyset) - \mu(\emptyset,\{1\}) - \mu(\emptyset,\{2\}) + \mu(\emptyset,\{1,2\})\\
I_\nu(\{1\},\{2\}) &= \mu(\{1\},\emptyset) - \mu(\emptyset,\emptyset) - \mu(\{1\},\{2\}) + \mu(\emptyset,\{2\})\\
I_\nu(\{2\},\{1\}) &= \mu(\{2\},\emptyset) - \mu(\emptyset,\emptyset) - \mu(\{2\},\{1\}) + \mu(\emptyset,\{1\})
\end{aligned}$$

When more criteria are involved, considering for instance the third interaction, we set

$$\Delta_{\{i\},\{j\}}\mu(A,B) = \mu(A\cup\{i\},B\cup\{j\}) - \mu(A\cup\{i\},B) - \mu(A,B\cup\{j\}) + \mu(A,B)$$

and

$$I_\mu(\{i\},\{j\}) = \sum_{K\subset N\setminus\{i,j\}} \frac{(n-|K|-2)!|K|!}{(n-1)!} \Delta_{\{i\},\{j\}}\mu\left(K, N\setminus(K\cup\{i,j\})\right) \qquad (6)$$

One can show that the general formula is [4]:

$$I_\mu(A,B) = \sum_{K\subset N\setminus(A\cup B)} \frac{(n-|K|-|A|-|B|)!|K|!}{(n-|A|-|B|+1)!} \Delta_{A,B}\mu(K, N\setminus(A\cup K)) \ . \qquad (7)$$

where $\Delta_{A,B}\mu(S,T) = \sum_{K\subset A, L\subset B}(-1)^{(|A|-|K|)+(|B|-|L|)}\mu(S\cup K, T\setminus L)$. $I_\mu(A,B)$ is the interaction index of μ on $A\cup B$ for attractive values of criteria A and repulsive values of criteria B. The interaction index is a key notion to define symmetric bi-capacities w.r.t. criteria P.

B. Bi-capacities with vanishing interaction indices

An interesting property for CPT-model (see Lemma 2) is that the associated bi-capacity (see (3)) satisfies $I_\mu(A,B) = 0$ for all $(A,B)\in\mathcal{Q}(N)$ with $A\neq\emptyset$ and $B\neq\emptyset$. This property is not surprising at all. Coming back to Section III-A, we have seen that the interaction index between i and j is zero if i and j are independent. This means that the contribution of the two criteria put together is just the sum of the elementary contributions of each criterion taken separately. Yet, from (3), the contribution in a CPT-model of the positive and the negative parts taken together is the difference between the elementary contributions of the positive and the negative criteria taken separately. Hence, it is natural to say that the positive criteria are *independent* to the negative criteria in the CPT model. This refers to a bipolar independence characterized by a vanishing bi-interaction index. Hence the property is intuitive.

We want to restrict this property to $N\setminus P$, leading to $I_\mu(A,B) = 0$ for all $(A,B)\in\mathcal{Q}(N)$ with $A\neq\emptyset$ and $B\neq\emptyset$. In particular, we obtain $I_\mu(\{i\},\{j\}) = 0$ for all $\{i,j\}\subset N\setminus P$. As we have seen with (5), if we restrict to a relation satisfied only on $N\setminus P$, the gain in terms of complexity will be quite negligible. So, we rather impose that all terms appearing in the expression of $I_\mu(\{i\},\{j\})$ vanish. We require thus by (6) that $\Delta_{\{i\},\{j\}}\mu(A,B) = 0$ for all $(A,B)\in\mathcal{Q}(N)$ with $A\neq\emptyset$ and $B\neq\emptyset$. Clearly, this property implies previous one.

Definition 1: Bi-capacity μ is said to be symmetric w.r.t. P (called P-symmetric) if $\Delta_{\{i\},\{j\}}\mu(A,B) = 0$ for all $(A,B)\in\mathcal{Q}(N\setminus\{i,j\})$ and all $\{i,j\}\subset N\setminus P$.

Let $\mathcal{G}_p^3(N)$ be the set of all bi-capacities defined on N that are P-symmetric w.r.t. to a coalition P of cardinality p. The following lemma can be shown:

Lemma 3: If μ is P-symmetric, then $I^\mu(A,B) = 0$ for all $(A,B)\in\mathcal{Q}(N)$ with $A\cap(N\setminus P)\neq\emptyset$ and $B\cap(N\setminus P)\neq\emptyset$. In particular, $I^\mu(\{i\},\{j\}) = 0$ whenever $\{i,j\}\subset N\setminus P$.

Let us see now the gain attained thanks to this definition. Set

$$\begin{aligned}\mathcal{Q}^P(N) \ &:= \ \{(A\cup A', B\cup B'), (A,B)\in\mathcal{Q}(P)\ , (A',B')\in\mathcal{Q}(N\setminus P)\\ &\qquad \text{with } A'=\emptyset \text{ or } B'=\emptyset\}\ .\end{aligned}$$

Let ν_P be the restriction of μ on $\mathcal{Q}^P(N)$. ν_P contains $3^{n-p}\times\left(2^{p+1}-1\right)$ terms. The next lemma shows that μ is determined only from a knowledge of ν_P, that is from μ on $\mathcal{Q}^P(N)$.

Lemma 4: If μ is P-symmetric, then for all $(A,B)\in\mathcal{Q}(P)$ and all $(A',B')\in\mathcal{Q}(N\setminus P)$

$$\mu(A\cup A', B\cup B') = \nu_P(A\cup A', B) + \nu_P(A, B\cup B') - \nu_P(A,B)\ .$$

It is interesting to investigate the two extreme cases. When $P = N$, we obtain $\mathcal{Q}^N(N) = \mathcal{Q}(N)$, so that usual bi-capacities are recovered. Now, let us look at the case $P=\emptyset$.

Lemma 5: A $\emptyset$-symmetric bi-capacity is equivalent to the CPT model.

Hence, P-symmetric bi-capacities derive from the CPT model in the sense that we obtain the CPT model when $P=\emptyset$. Thus we obtain a definition ranging from the CPT-models to usual bi-capacities. Moreover, the number of unknowns is what we wanted.

C. Comparison of the complexity of the models

Table I below gives a comparison of the number of unknowns obtained in each model. For P-symmetric bi-capacities, we consider the case where P is a singleton. The complexity of $G_1^3(N)$ is $3/2$ times as much as that of the CPT model $G_{\pm}^2(N)$, and the complexity of $G_1^3(N)$ is roughly 3 times as much as that of $G^2(N)$. We see that the gain in using $G_1^3(N)$ instead of $G_1^3(N)$ is quite significative.

model	complexity	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$\mathcal{G}^2(N)$	2^n	8	16	32	64	128
$\mathcal{G}_{\pm}^2(N)$	2^{n+1}	16	32	64	128	256
$\mathcal{G}_1^3(N)$	$3\times(2^n-1)$	21	45	93	183	381
$\mathcal{G}^3(N)$	3^n	27	81	273	819	2187

TABLE I

COMPARISON OF THE COMPLEXITY OF THE MODELS.

IV. AN ILLUSTRATIVE EXAMPLE

Let us conclude this paper by applying the concept of P-symmetric bi-capacity on an example.

A. Description of the example

The director of a University decides on students who are applying for graduate studies in economics where some prerequisites from school are required. Students are indeed evaluated according to mathematics (M), statistics (S) and language skills (L). All the marks with respect to the scores are given on the same bipolar scale from -10 to 10 with neutral value 0. These three criteria serve as a basis for a pre-selection of the candidates. The best candidates have then an interview with a jury composed of members of the University to assess their motivation in studying economics. In general, the applicants have generally speaking a strong scientific background so that mathematics and statistics have a big importance to the director. However, he does not wish to favor too much students that have a scientific profile with some flaws in languages. Besides, mathematics and statistics are in some sense *substituable*, since, usually, students good at mathematics are also good at statistics. As a consequence, comparing two students good in mathematics, the director prefers the one that is better in languages even if he is worst in statistics. Consider the following students :

	M	S	L
student A	4	6	-3
student B	4	5	-2

Student A is highly penalized by his performance in languages. Henceforth, the director would prefer a student (with the same mark in mathematics) that is a little bit better in languages even if the student would be a little bit worse in statistics. This means that the director prefers B to A:

$$A \prec B \tag{8}$$

Consider now a student that has a weakness in mathematics. In this case, since the applicants are supposed to have strong scientific skills, a student good in statistics is now preferred to one good in languages. Hence, the following two students

	M	S	L
student C	-1	6	-3
student D	-1	5	-2

are ranked as follows :

$$C \succ D \tag{9}$$

B. Attempt to model previous example with a bi-capacity

(8) and (9) cannot be modeled with the CPT model. Let us try to model (8) and (9) with the extension of the Choquet integral to bi-capacities. We have $BC_\mu(4,6,-3) = 3\mu(\{M,S\},\{L\}) + \mu(\{M,S\},\emptyset) + 2\mu(\{S\},\emptyset)$ and $BC_\mu(4,5,-2) = 2\mu(\{M,S\},\{L\}) + 2\mu(\{M,S\},\emptyset) + \mu(\{S\},\emptyset)$. Hence (8) is equivalent to

$$\mu(\{M,S\},\emptyset) - \mu(\{M,S\},\{L\}) > \mu(\{S\},\emptyset)$$

Similarly, relation (9) is equivalent to

$$\mu(\{S\},\{L\}) > 0.$$

There is no contradiction between theses two inequalities. Henceforth, BC_μ is able to model the example. This aggregation operator models the expertise that makes an explicit reference to an absolute value.

C. Attempt to model previous example with P-symmetric bi-capacities

Let us try to model (8) and (9) with a bi-capacity with vanishing interaction indices. As in previous section, we consider $P = \{M\}$.

$$\mathcal{Q}^{\{M\}}(\{M,S,L\}) \ := \ \{(A\cup A', B\cup B'), (A,B)\in\mathcal{Q}(\{M\}), (A',B') \in$$
$$\text{with } A'=\emptyset \text{ or } B'=\emptyset\} \ .$$

We consider $\mu_{\{M\}}$. Hence by Lemma 4

$$\begin{aligned} A \prec B &\iff \mu(\{M,S\},\emptyset) - \mu(\{M,S\},\{L\}) > \mu(\{S\},\emptyset) \\ &\iff -\nu_M(\{M\},\{S\}) + \nu_M(\{M\},\emptyset) > \nu_M(\{S\},\emptyset) \end{aligned}$$

Similarly,

$$\begin{aligned} C \succ D &\iff \mu(\{S\},\{L\}) > 0 \\ &\iff \nu_M(\{S\},\emptyset) + \nu_M(\emptyset,\{L\}) > 0 \ . \end{aligned}$$

There is no contradiction between previous two relations. A $\{M\}$-symmetric bi-capacity is thus able to model previous example.

When $P=\emptyset$, we obtain

$$A \prec B \iff -\nu_\emptyset(\emptyset,\{L\}) > \nu_\emptyset(\{S\},\emptyset)$$

and

$$C \succ D \iff \nu_\emptyset(\emptyset,\{L\}) + \nu_\emptyset(\{S\},\emptyset) > 0$$

These two relations are contradictory, which is not surprising since this model is equivalent to the CPT model.

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