New characterizations of k-additivity and k-monotonicity of bi-capacities

Katsushige FUJIMOTO
Faculty of Economics, Fukushima University,
1 Kanayagawa Fukushima, 960-1296, Japan,
Email: fujimoto@econ.fukushima-u.ac.jp

***Abstract*— This paper first proposes an extension of dividends of cooperative games to that of bi-capacities (bi-cooperative games) defined on 3^N, which is not the Möbius transform. k-Additivity of bi-capacities proposed by Grabisch and Labreuche [9] is re-characterized through the use of the extended dividends. The notion of k-monotonicity of bi-capacities is also characterized through each of the following notions: the extended dividends, (S,T)-discrete derivatives, and $|S \cup T|$-order derivatives of the piecewise multilinear extension of the ternary pseudo-Boolean function corresponding to the capacities.**

I. Introduction

Let $N = \{1, \ldots, n\}$ be a set of criteria describing the preference of a decision maker over a set U of objects in a multicriteria decision problem, or a set of players in a cooperative game, etc... Capacities [4], set functions which vanish on the empty set and be monotone (1-monotone) w.r.t. set-inclusion, have become a fundamental tool in decision making, especially decision under uncertainty and decision making under multiple criteria. Characteristic functions, in cooperative game theory, are set functions vanishing on the empty set. These concepts are defined on 2^N.

Recently, there have been some attempt to define more general concept in game theory and multicriteria decision theory. In game theory, Aubin [1] has proposed the concept of *generalized coalition* as a function $c : N \to [-1, 1]$ which associates each player i with his/her level of participation $c(i) \in [-1, 1]$. A positive level is interpreted as attraction of the player i for the coalition, and a negative level as repulsion. Later, Bilbao *et. al.* [2] have proposed what they call *bicooperative games*, which generalize the idea of *ternary voting games* proposed by Felsenthal and Machover [6] on the set of all *signed coalitions* given by $\{c : N \to \{-1, 0, 1\}\}$. (Recall that the set 2^N is corresponding to the set of all functions $c : N \to \{0, 1\}$). Here, $\{c : N \to \{-1, 0, 1\}\} \cong \{(S,T) \mid S,T \subseteq N, S \cap T = \emptyset\} = 3^N$. In multicriteria decision theory, Grabisch and Labreuche [9] have proposed what they call *bi-capacities* on the lattice $(3^N, \sqsubseteq)$, which are monotone functions w.r.t. the order $\sqsubseteq$ on 3^N, where $(S_1, S_2) \sqsubseteq (T_1, T_2)$ if and only if $S_1 \subseteq T_1$ and $S_2 \supseteq T_2$. (Recall that the set 2^N can be regarded as the lattice $(2^N, \subseteq)$ equipped with the order $\subseteq$ induced by set-inclusion, and ordinary capacities are monotone functions w.r.t. $\subseteq$ on 2^N). Moreover, they have proposed the Möbius transform [9] and the concept of k-*additivity* [9] of bi-capacities via the order $\sqsubseteq$ on 3^N, and the concept of k-*monotonicity* [11] of bi-capacities as an extension of that of capacities.

This paper proposes an extension of dividends of cooperative games to that of bi-capacities (bi-cooperative games) defined on 3^N, which is not the Möbius transform of the bi-capacities (The dividends in an ordinary cooperative game correspond to the Möbius transform of the game). New characterizations of k-additivity and k-monotonicity of bi-capacities are stated through the use of the generalized dividends.

Throughout this paper, to avoid heavy notations, we will often omit braces to denote singletons.

II. Bi-capacities

We denote $Q(N) := \{(A_1, A_2) \in \mathcal{P}(N) \times \mathcal{P}(N) | A_1 \cap A_2 = \emptyset\} = 3^N$, where $\mathcal{P}(N)$ stands for 2^N. When equipped with the following order: for $(A_1, A_2), (B_1, B_2) \in Q(N)$

$$(A_1, A_2) \sqsubseteq (B_1, B_2) \text{ iff } A_1 \subseteq B_1 \text{ and } A_2 \supseteq B_2,$$

$(Q(N), \sqsubseteq)$ is the lattice 3^N. Sup and inf are given by

$$(A_1, A_2) \sqcup (B_1, B_2) = (A_1 \cup B_1, A_2 \cap B_2),$$
$$(A_1, A_2) \sqcap (B_1, B_2) = (A_1 \cap B_1, A_2 \cup B_2).$$

Top and bottom are respectively $(N, \emptyset)$ and $(\emptyset, N)$.

Definition 1 (irreducible elements [5]*):*
Let $(L, \leq, \vee, \wedge, \top, \bot)$ be a lattice. An element $x \in L$ is $\vee$-*irreducible* if $x \neq \bot$ and $x = a \vee b$ implies $x = a$ or $x = b$, $\forall a, b \in L$, where $\vee, \wedge, \top, \bot$ denotes $\sup, \inf$, the top and bottom element, respectively.

Proposition 1: [9] The $\sqcup$-irreducible elements of $Q(N)$ are $(\emptyset, i^c)$ and (i, i^c), for all $i \in N$. Moreover, for any $(A_1, A_2) \in Q(N)$,

$$(A_1, A_2) = \bigsqcup_{i \in A_1} (i, i^c) \quad \sqcup \bigsqcup_{j \in N \setminus (A_1 \cup A_2)} (\emptyset, j^c). \qquad (1)$$

The equation (1) is called the *minimal decomposition* of (A_1, A_2).

Note [9]: $\sqcup$-irreducible elements permit to define *layer* in $Q(N)$ as follows: $(\emptyset, N)$ is the bottom layer (layer 0) (the black square on Fig.1), the set of all $\sqcup$-irreducible elements forms layer 1 (black circles on Fig.1), and layer k, for $k = 2, \ldots n$, contains all elements whose minimal decomposition contains exactly k $\sqcup$-irreducible elements. In other words, layer k contains all elements $(A_1, A_2) \in Q(N)$ such that $|A_2| = n - k$, for $k = 2, \ldots, n$. On the other hand, we consider another lattice $(\mathcal{P}(N), \subseteq, \cup, \cap, N, \emptyset)$. Then, the empty set is the bottom layer; all singletons are $\cup$-irreducible elements, (i.e. in layer 1); the set of all $A \in \mathcal{P}(N)$ whose cardinality is k, for $k = 2, \ldots, n$, forms layer k.

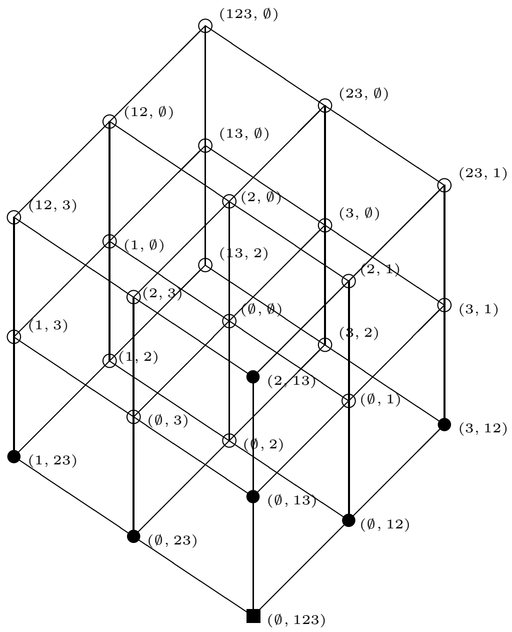

Fig. 1. The lattice Q(123): the element in layer 0 is indicated by a black square and elements in layer 1 black circles.

Definition 2 (bi-capacity [9]): A function $v : Q(N) \to \mathbb{R}$ is a bi-capacity if it satisfies:

(i) $v(\emptyset, \emptyset) = 0$,

(ii) $A \sqsubseteq B, A, B \in Q(N)$ implies $v(A) \leq v(B)$.

In addition, v is *normalized* if $v(N, \emptyset) = 1 = -v(\emptyset, N)$; v is called additive if it satisfies for all $(A_1, A_2) \in Q(N)$,

$$v(A_1, A_2) = \sum_{i \in A_1} v(i, \emptyset) + \sum_{j \in A_2} v(\emptyset, j); \qquad (2)$$

v is said to be monotone (1-monotone) if it satisfies this condition (ii).

Definition 3 (Möbius transform of bi-capacity [9]): To any bi-capacity $v : Q(N) \to \mathbb{R}$, another function $m : Q(N) \to \mathbb{R}$ can be associated by

$$m(A_1, A_2) := \sum_{\substack{B_1 \subseteq A_1 \\ A_2 \subseteq B_2 \subseteq A_1{}^c}} (-1)^{|A_1 \setminus B_1| + |B_2 \setminus A_2|} v(B_1, B_2) \qquad (3)$$

for $(A_1, A_2) \in Q(N)$. This correspondence proves to be one-to-one, since conversely

$$v(A_1, A_2) = \sum_{(B_1, B_2) \sqsubseteq (A_1, A_2)} m(B_1, B_2) \qquad (4)$$

for all $(A_1, A_2) \in Q(N)$. The validity of (4) is proven by Grabisch and Labreuche [9] who call m the Möbius transform of a bi-capacity v.

Definition 4: To any bi-capacity $v : Q(N) \to \mathbb{R}$, another function $d : Q(N) \to \mathbb{R}$ can be associated by

$$d(A_1, A_2) := \sum_{\substack{B_1 \subseteq A_1 \\ B_2 \subseteq A_2}} (-1)^{|A_1 \setminus B_1| + |A_2 \setminus B_2|} v(B_1, B_2) \qquad (5)$$

$$= \sum_{(\emptyset, A_2) \sqsubseteq (B_1, B_2) \sqsubseteq (A_1, \emptyset)} (-1)^{|A_1 \setminus B_1| + |A_2 \setminus B_2|} v(B_1, B_2)$$

for $(A_1, A_2) \in Q(N)$.

Proposition 2: Let $v : Q(N) \to \mathbb{R}$ be a bi-capacity, and $d : Q(N) \to \mathbb{R}$ the function defined by (5). Then,

$$v(A_1, A_2) = \sum_{(\emptyset, A_2) \sqsubseteq (B_1, B_2) \sqsubseteq (A_1, \emptyset)} d(B_1, B_2) \qquad (6)$$

$$= \sum_{\substack{B_1 \subseteq A_1 \\ B_2 \subseteq A_2}} d(B_1, B_2) \qquad (7)$$

for all $(A_1, A_2) \in Q(N)$.

Property 1: Let $v : Q(N) \to \mathbb{R}$ be a bi-capacity, $m : Q(N) \to \mathbb{R}$ the Möbius transform of v, $d : Q(N) \to \mathbb{R}$ the function defined by (5), $\mu : \mathcal{P}(N) \to \mathbb{R}$ a capacity, and $m^\mu : \mathcal{P}(N) \to \mathbb{R}$ the Möbius transform of μ. If $v(A_1, A_2) = \mu(A_1)$ for all $(A_1, A_2) \in Q(N)$, then

$$m(A_1, A_2) = d(A_1, A_2) = m^\mu(A_1)$$

for all $(A_1, A_2) \in Q(N)$.

Property 2: Let $v : Q(N) \to \mathbb{R}$ be a bi-capacity. If there exist two capacities $\mu_1, \mu_2 : \mathcal{P}(N) \to \mathbb{R}$ such that

$$v(A_1, A_2) = \mu_1(A_1) - \mu_2(A_2), \quad \forall (A_1, A_2) \in Q(N),$$

then

$$m(A_1, A_2) = \begin{cases} m^{\mu_1}(A_1) & \text{if } A_1 = (A_2)^c, A_1 \neq \emptyset \\ m^{\overline{\mu_2}}({A_2}^c) & \text{if } A_1 = \emptyset, A_2 \subsetneq N \\ m^{\mu_2}(A_2) & \text{if } A_1 = \emptyset, A_2 = N \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{\mu_2}(A) = \mu_2(N) - \mu_2(N \setminus A)$ [9], and

$$d(A_1, A_2) = \begin{cases} m^{\mu_1}(A_1) & \text{if } A_2 = \emptyset \\ -m^{\mu_2}(A_2) & \text{if } A_1 = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Property 3: Let $v : Q(N) \to \mathbb{R}$ be a bi-capacity. If v is additive,

$$v(A_1, A_2) = \sum_{i \in A_1} m(i, i^c) + \sum_{j \in A_2} m(\emptyset, j^c) \quad [9]$$

and

$$v(A_1, A_2) = \sum_{i \in A_1} d(i, \emptyset) + \sum_{j \in A_2} d(\emptyset, j)$$

for all $(A_1, A_2) \in Q(N)$.

The next proposition shows a relation between m and d.

Proposition 3: Let $v : Q(N) \to \mathbb{R}$ be a bi-capacity, $m : Q(N) \to \mathbb{R}$ the Möbius transform of v, and $d : Q(N) \to \mathbb{R}$ the function defined by (5). Then

$$m(A_1, A_2) = (-1)^{|{A_1}^c \setminus A_2|} \sum_{{A_1}^c \setminus A_2 \subseteq C_2 \subseteq {A_1}^c} d(A_1, C_2) \quad (8)$$

and

$$d(A_1, A_2) = (-1)^{|A_2|} \sum_{C_2 \subseteq {A_1}^c \setminus A_2} m(A_1, C_2) \quad (9)$$

for all $(A_1, A_2) \in Q(N)$.

III. k-ADDITIVITY

Grabisch and Labreuche [9] introduced the concept of k-additivity into bi-capacities.

Definition 5 (k-additivity of capacity [7]*):* Given an integer $k \in \{1, \ldots, n-1\}$, a capacity μ is said to be *k-additive* if $m^{\mu}(A) = 0$ whenever $|A| > k$.

Definition 6 (k-additivity of bi-capacity[9]*):* Given an integer $k \in \{1, \ldots, n-1\}$, a bi-capacity is said to be *k-additive* if its Möbius transform vanishes for all elements in layer l whenever $l > k$. Equivalently, a bi-capacity v is k-additive if and only if $m(A_1, A_2) = 0$ whenever $|{A_2}^c| > k$.

Proposition 4: Let $v : Q(N) \to \mathbb{R}$ be a bi-capacity, $m : Q(N) \to \mathbb{R}$ the Möbius transform of v, and $d : Q(N) \to \mathbb{R}$ the function defined by (5). Given an integer $k \in \{1, \ldots, n-1\}$, the following three conditions are equivalent to each other.

(i) v is k-additive.

(ii) $m(A_1, A_2) = 0$ if $|{A_2}^c| > k$.

(iii) $d(A_1, A_2) = 0$ if $|A_1 \cup A_2| > k$.

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For example, a bi-capacity is 1-additive if its Möbius transform (resp. $d : Q(N) \to \mathbb{R}$) vanishes for all elements indicated by open circles on Fig. 1 (resp. Fig. 2).

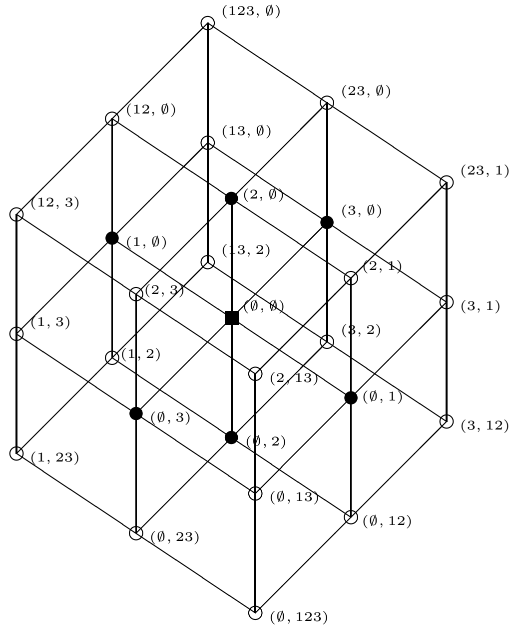

Fig. 2. The lattice Q(123)

Definition 7 ($\mathcal{C}$-decomposition of bi-capacity): Let v be a bi-capacity on $Q(N)$ and $\mathcal{C} \subseteq \mathcal{P}(N)$ a covering of N (i.e. $\bigcup_{C \in \mathcal{C}} C = N$). A family $\{v_C\}_{C \in \mathcal{C}}$ is called a *$\mathcal{C}$-decomposition* of v if each v_C is a function on $Q(C)$ vanishing on $(\emptyset, \emptyset)$ and the following holds:

$$v(A_1, A_2) = \sum_{C \in \mathcal{C}} v_C(A_1 \cap C, A_2 \cap C) \quad (10)$$

for all $(A_1, A_2) \in Q(N)$.

Proposition 5: Let $v : Q(N) \to \mathbb{R}$ be a bi-capacity, $m : Q(N) \to \mathbb{R}$ the Möbius transform of v, and $d : Q(N) \to \mathbb{R}$ the function defined by (5). Then, given a subset C of N,

$$m(A_1, A_2) = 0 \text{ whenever } {A_2}^c \supsetneq C$$

if and only if

$$d(B_1, B_2) = 0 \text{ whenever } B_1 \cup B_2 \supsetneq C;$$

moreover,

$$m(A_1, A_2) = 0 \text{ whenever } {A_2}^c \nsubseteq C$$

if and only if

$$d(B_1, B_2) = 0 \text{ whenever } B_1 \cup B_2 \nsubseteq C.$$

Proposition 6: Let $v : Q(N) \to \mathbb{R}$ be a bi-capacity, $m : Q(N) \to \mathbb{R}$ the Möbius transform of v, $d : Q(N) \to \mathbb{R}$ the function defined by (5), and $\mathcal{C}$ a covering of N. Then, the following three conditions are equivalent to each other.

(i) v has a $\mathcal{C}$-decomposition.

(ii) $m(A_1, A_2) = 0$ if ${A_2}^c \nsubseteq C$ for all $C \in \mathcal{C}$.

(iii) $d(A_1, A_2) = 0$ if $(A_1 \cup A_2) \nsubseteq C$ for all $C \in \mathcal{C}$.

Clearly, the bi-capacity v having $\mathcal{C}$-decomposition is k-additive, where $k = \max_{C\in\mathcal{C}}\{|C|\}$.

Proposition 7: Let $\mathcal{C} \subseteq \mathcal{P}(N)$ is a covering of N. A bi-capacity v on $Q(N)$ has a $\mathcal{C}$-decomposition if and only if

$$v(A_1, A_2) = \sum_{\emptyset \neq \mathcal{D} \subseteq \mathcal{C}} (-1)^{|\mathcal{D}|+1} v(\bigcap_{D\in\mathcal{D}} D \cap A_1, \bigcap_{D\in\mathcal{D}} D \cap A_2)$$

for all $(A_1, A_2) \in Q(N)$.

IV. Ternary pseudo-Boolean functions and The Choquet integral

Recall that $\{v : Q(N) \to \mathbb{R}\} \cong \{f : \{-1, 0, 1\}^N \to \mathbb{R}\}$. Then, for any bi-capacity $v : Q(N) \to \mathbb{R}$ there exists a ternary pseudo-Boolean function f_v (i.e. $f_v : \{-1, 0, 1\}^N \to \mathbb{R}$) corresponding to v. Now, we propose equivalent representations, by using ternary pseudo-Boolean functions, of v as follows:

$$f_v(\boldsymbol{x}) := \sum_{(S_1,S_2)\in Q(N)} d(S_1, S_2) \bigwedge_{i\in S_1} (x_i \vee 0) \wedge \bigwedge_{j\in S_2} (-x_i \vee 0) \quad (11)$$

$$= \sum_{(S_1,S_2)\in Q(N)} d(S_1, S_2) \prod_{i\in S_1} (x_i \vee 0) \times \prod_{j\in S_2} (-x_i \vee 0) \quad (12)$$

for $\boldsymbol{x} \in \{-1, 0, 1\}^N$. This correspondence is represented as

$$f_v(e_{(A_1,A_2)}) = v(A_1, A_2) \quad \forall (A_1, A_2) \in Q(N),$$

where $e_{(A_1,A_2)}$ denotes a characteristic vector of (A_1, A_2), which is the vector of $\{-1, 0, 1\}^N$ whose i-th element is 1 if $i \in A_1$, -1 if $i \in A_2$, and 0 otherwise.

Equation (11) leads to the Choquet integral w.r.t. a bi-capacity (This fact will be shown in Proposition 10). On the other hand, Equation (11) leads to the piecewise multilinear extension of the ternary pseudo-Boolean function corresponding to the bi-capacity as follows:

Definition 8 (piecewise multilinear extension): Let v be a bi-capacity and f_v a ternary pseudo-Boolean function corresponding to v. The *piecewise multilinear extension* of f_v is a real-valued function g_v on $[-1, 1]^N$ defined by

$$g_v(\boldsymbol{x}) = \sum_{(S_1,S_2)\in Q(N)} d(S_1, S_2) \prod_{i\in S_1} (x_i \vee 0) \times \prod_{j\in S_2} (-x_i \vee 0) \quad (13)$$

for $\boldsymbol{x} \in [-1, 1]^N$.

Let $\mathcal{F}(A_1, A_2) := \{\boldsymbol{x} \in \{-1, 0, 1\}^N \mid x_i \geq 0 \ \forall i \in A_1 \ and \ x_j \leq 0 \ \forall j \in A_2\}$ and $\mathcal{G}(A_1, A_2) := \{\boldsymbol{x} \in [-1, 1]^N \mid x_i \geq 0 \ \forall i \in A_1 \ and \ x_j \leq 0 \ \forall j \in A_2\}$ for $(A_1, A_2) \in Q(N)$. Then, $g_v|_{\mathcal{G}(A_1,A_2)}$ is the unique multilinear function on $\mathcal{G}(A_1, A_2)$ such that $g_v|_{\mathcal{G}(A_1,A_2)}(\boldsymbol{x}) = f(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathcal{F}(A_1, A_2)$, where $g_v|_{\mathcal{G}(A_1,A_2)}$ is the restriction of g to $\mathcal{G}(A_1, A_2)$. Moreover, $g_v|_{\mathcal{G}(A_1,A_2)}$ is expressed as follows:

$$g_v|_{\mathcal{G}(A_1,A_2)}(\boldsymbol{x}) = \sum_{\substack{B_1\subseteq A_1\\ B_2\subseteq A_2}} d(B_1, B_2) \prod_{i\in B_1} x_i \times \prod_{j\in B_2} (-x_j)$$

for $\boldsymbol{x} \in \mathcal{G}(A_1, A_2)$, and $g_v|_{\mathcal{G}(A_1,A_2)}(\boldsymbol{x}) = g_v|_{\mathcal{G}(B_1,B_2)}(\boldsymbol{x})$ $\forall \boldsymbol{x} \in \mathcal{G}(A_1, A_2) \cap \mathcal{G}(B_1, B_2)$ if $\mathcal{G}(A_1, A_2) \cap \mathcal{G}(B_1, B_2) \neq \emptyset$. Therefore, the g_v is a piecewise multilinear function.

Definition 9 (Choquet integral w.r.t. capacity [4]*):* Let μ be a capacity and f a non-negative function defined on N. The Choquet integral $C_\mu(f)$ of f w.r.t. μ is defined by

$$C_\mu(f) := \sum_{i=1}^{n} (f_{\sigma(i)} - f_{\sigma(i-1)}) \ \mu(A_i),$$

where σ is a permutation on N such that $0 = f_{\sigma(0)} \leq f_{\sigma(1)} \leq \cdots \leq f_{\sigma(n)}$, $A_i := \{\sigma(i), \ldots, \sigma(n)\}$, and f_i denotes the value $f(i)$ for each $i \in N$.

An important remark is that μ needs not to be a capacity (i.e. monotone) in order to that the Choquet integral is properly defined: any set function vanishing on the empty set could do.

The expression of the Choquet integral w.r.t. bi-capacity has been introduced axiomatically in Labreuche and Grabisch [12].

Definition 10 (Choquet integral w.r.t. bi-capacity [12]*):* Let v be a bi-capacity and f a real-valued function on N. The Choquet integral $C_v(f)$ of f w.r.t. v is given by

$$C_v(f) := C_{\mu_{N^+}}(|f|),$$

where μ_{N^+} is a real-valued set function on $\mathcal{P}(N)$ defined by $\mu_{N^+}(C) = v(C \cap N^+, C \cap N^-)$, and $N^+ := \{i \in N | f_i \geq 0\}$, $N^- := N \setminus N^+$, and f_i denotes the value $f(i)$ for each $i \in N$.

It should be noticed that μ_{N^+} is not a capacity in general, since it may be not monotone, and even take negative values. Observe that we have $C_v(1_{(A_1,A_2)}) = v(A_1, A_2)$ for any $(A_1, A_2) \in Q(N)$, where $1_{(A_1,A_2)}$ is the characteristic function of (A_1, A_2). Hence, the Choquet integral is indeed an extension of v.

Proposition 8: [14] Let μ be a capacity on $\mathcal{P}(N)$. Then,

$$C_\mu(f) = \sum_{S \in \mathcal{P}(N)} m^\mu(S) \bigwedge_{i \in S} f_i$$

for all $f : N \to \mathbb{R}^+$, where f_i denotes the value $f(i)$ for each $i \in N$.

Proposition 9: [10] Let v be a bi-capacity on $Q(N)$. Then,

$$\begin{aligned} C_v(f) &= \sum_{B \subseteq N} m(\emptyset, B) \bigwedge_{i \in B^c \cap N^-} f_i \\ &+ \sum_{\substack{(S_1,S_2) \in Q(N) \\ S_1 \neq \emptyset}} m(S_1, S_2) \cdot \\ &\quad \left[\left(\bigwedge_{i \in (S_1 \cup S_2)^c \cap N^-} f_i + \bigwedge_{j \in S_1} f_j \right) \vee 0 \right] \end{aligned}$$

for all $f : N \to \mathbb{R}$, where f_i denotes the value $f(i)$ for each $i \in N$.

Proposition 10: Let v be a bi-capacity on $Q(N)$. Then,

$$C_v(f) = \sum_{(S_1,S_2) \in Q(N)} d(S_1, S_2) \bigwedge_{i \in S_1} (f_i \vee 0) \wedge \bigwedge_{j \in S_2} (-f_j \vee 0)$$

for all $f : N \to \mathbb{R}$, where f_i denotes the value $f(i)$ for each $i \in N$.

V. Derivatives of Bi-capacity

Grabisch and Labreuche [9] extended the notion of *k-order derivative* of capacities to that of bi-capacities.

Definition 11 (T-derivative of capacity [7]*):* Let μ be a capacity defined on $\mathcal{P}(N)$ and $T \in \mathcal{P}(N)$. The T-derivative at a point $S \in \mathcal{P}(N \setminus T)$, is denoted as $\Delta_T v(S \cup T)$ and defined by

$$\Delta_T v(S \cup T) := \sum_{L \subseteq T} (-1)^{|T \setminus L|} v(S \cup L). \tag{14}$$

Definition 12 ((T_1, T_2)-derivative of bi-capacity [10]*):* Let v be a bi-capacity defined on $Q(N)$ and $(T_1, T_2) \in Q(N)$. The (T_1, T_2)-derivative at a point $(S_1, S_2 \cup T_2)$, $(S_1, S_2) \in Q(N \setminus (T_1 \cup T_2))$, is denoted as $\Delta_{(T_1,T_2)} v(S_1 \cup T_1, S_2 \cup T_2)$ and defined by

$$\begin{aligned} &\Delta_{(T_1,T_2)} v(S_1 \cup T_1, S_2 \cup T_2) \\ &\quad := \sum_{\substack{L_1 \subseteq T_1 \\ L_2 \subseteq T_2}} (-1)^{|T_1 \setminus L_1| + |L_2|} v(S_1 \cup L_1, S_2 \cup L_2). \end{aligned} \tag{15}$$

The formula (14) is led by the following recursive relations [10]: $\Delta_i v(S \cup i) := v(S \cup i) - v(S)$ and $\Delta_T v(S \cup T)$ $:= \Delta_i \left(\Delta_{T \setminus i} v(S \cup T) \right)$, $i \in T$; the formula (15) is led by the following recursive relations [10]:

$$\Delta_{(i,\emptyset)} v(S_1 \cup i, S_2) := v(S_1 \cup i, S_2) - v(S_1, S_2),$$

$$\Delta_{(\emptyset,j)} v(S_1, S_2 \cup j) := v(S_1, S_2) - v(S_1, S_2 \cup j),$$

$$\begin{aligned} &\Delta_{(T_1,T_2)} v(S_1 \cup T_1, S_2 \cup T_2) \\ &\quad := \Delta_{(i,\emptyset)} \left(\Delta_{(T_1 \setminus i, T_2)} v(S_1 \cup T_1, S_2 \cup T_2) \right) \\ &\quad = \Delta_{(\emptyset,j)} \left(\Delta_{(T_1, T_2 \setminus j)} v(S_1 \cup T_1, S_2 \cup T_2) \right), \\ &i \in T_1, j \in T_2. \end{aligned}$$

Proposition 11: [9] For $(T_1, T_2) \in Q(N)$,

$$\begin{aligned} &\Delta_{(T_1,T_2)} v(S_1 \cup T_1, S_2 \cup T_2) \\ &= \sum_{(T_1, N \setminus (T_1 \cup T_2)) \sqsubseteq (A_1,A_2) \sqsubseteq (S_1 \cup T_1, S_2)} m(A_1, A_2) \end{aligned} \tag{16}$$

for any $(S_1, S_2) \in Q(N \setminus (T_1 \cup T_2))$.

Proposition 12: For $(T_1, T_2) \in Q(N)$,

$$\begin{aligned} &\Delta_{(T_1,T_2)} v(S_1 \cup T_1, S_2 \cup T_2) \\ &\quad = \sum_{\substack{L_1 \subseteq S_1 \\ L_2 \subseteq S_2}} (-1)^{|L_2|} d(L_1 \cup T_1, L_2 \cup T_2) \end{aligned} \tag{17}$$

for any $(S_1, S_2) \in Q(N \setminus (T_1 \cup T_2))$.

Proposition 13: Let v be a bi-capacity and $g_v : [-1, 1]^N \to \mathbb{R}$ the piecewise multilinear extension of the ternary pseudo-Boolean function corresponding to v, which is defined in *definition 8*, and $(T_1, T_2) := (\{t_1^1, \ldots, t_1^i\}, \{t_2^1, \ldots, t_2^j\}) \in Q(N)$. Then,

$$\begin{aligned} &\Delta_{(T_1,T_2)} v(S_1 \cup T_1, S_2 \cup T_2) \\ &\quad = \frac{\partial^{(i+j)}}{\partial t_1^1, \ldots, \partial t_1^i \partial t_2^1, \ldots, \partial t_2^j} g_v(e_{(S_1 \cup T_1, S_2 \cup T_2)}) \end{aligned} \tag{18}$$

for all $(S_1, S_2) \in Q(N \setminus (T_1 \cup T_2))$, where $e_{(S_1 \cup T_1, S_2 \cup T_2)} \in [-1, 1]^N$ is the characteristic vector of $(S_1 \cup T_1, S_2 \cup T_2) \in Q(N)$.

VI. k-MONOTONICITY

Labreuche and Grabisch [11] extended the notion of k-monotonicity of capacities to that of bi-capacities.

Definition 13 (k-monotonicity of capacity [3]*):* Given an integer $k \geq 2$, a capacity μ on $\mathcal{P}(N)$ is said to be k-monotone if and only if

$$\mu\left(\bigcup_{i=1}^{k} S_i\right) \geq \sum_{\substack{I \subseteq \{1,\dots,k\} \\ I \neq \emptyset}} (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} S_i\right) \quad (19)$$

for any $S_i \in \mathcal{P}(N)$, $2 \leq i \leq k$.

Definition 14 (k-monotonicity of bi-capacity [11]*):* Given an integer $k \geq 2$, a bi-capacity v on $Q(N)$ is said to be k-monotone if and only if

$$v\left(\bigsqcup_{i=1}^{k} S_i\right) \geq \sum_{\substack{I \subseteq \{1,\dots,k\} \\ I \neq \emptyset}} (-1)^{|I|+1} v\left(\sqcap_{i \in I} S_i\right) \quad (20)$$

for any $S_i \in Q(N)$, $2 \leq i \leq k$.

It is easy to verify that k-monotonicity, with any $k \geq 2$, implies l-monotonicity for all $2 \leq l \leq k$. By extension, 1-monotonicity (which does not correspond to $k = 1$ in each of Equations (19) and (20)) is defined as standard monotonicity.

Proposition 14: [3] Let μ be a capacity defined on $\mathcal{P}(N)$. For a given integer $k \geq 1$, the following three conditions are equivalent to each other:

(i) μ is k-monotone.

(ii) $\sum_{C \subseteq B \subseteq A} m^{\mu}(B) \geq 0$
for all $A \in \mathcal{P}(N)$ and $C \in \mathcal{P}(N)$ with $1 \leq |C| \leq k$.

(iii) $\Delta_T v(S \cup T) \geq 0$
for all $T \in \mathcal{P}(N)$ with $1 \leq |T| \leq k$ and all $S \in \mathcal{P}(N \setminus T)$.

Proposition 15: Let v be a bi-capacity defined on $Q(N)$. For a given integer $k \geq 1$, the following four conditions are equivalent to each other:

(i) v is k-monotone.

(ii) $$\sum_{(C_1,C_2) \sqsubseteq (B_1,B_2) \sqsubseteq (A_1,A_2)} m(B_1, B_2) \geq 0$$
for all $(A_1, A_2) \in Q(N)$ and all $(C_1, C_2) \in Q(N)$ with $1 \leq |{C_2}^c| \leq k$.

(iii) $$\sum_{\substack{C_1 \subseteq B_1 \subseteq A_1 \\ C_2 \subseteq B_2 \subseteq A_2}} (-1)^{|C_2|} d(B_1, B_2) \geq 0$$
for all $(A_1, A_2) \in Q(N)$ and all $(C_1, C_2) \in Q(N)$ with $1 \leq |C_1 \cup C_2| \leq k$.

(iv) $\Delta_{(T_1,T_2)} v(S_1 \cup T_1, S_2 \cup T_2) \geq 0$
for all $(T_1, T_2) \in Q(N)$ with $1 \leq |T_1 \cup T_2| \leq k$ and all $(S_1, S_2) \in Q(N \setminus (T_1 \cup T_2))$.

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