On distorted probabilities and m-dimensional distorted probabilities: A review

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Abstract

Although fuzzy measures are powerful tools in economics, decision theory and artificial intelligence, their use in practical applications is more difficult due to the fact that they require a large number of parameters. To reduce such number of parameters several approaches have been considered in the literature. One of them is the so-called distorted probabilities or, equivalently, fuzzy measures that can be represented in terms of a probability distribution and a non-decreasing function (a distortion function). In this work we review some results for distorted probabilities and m-dimensional distorted probabilities that have been recently proposed. We also present a new result about m-dimensional probabilities.

Keywords : Fuzzy measure, Non-additive measure, Distorted probability, m-dimensional distorted probability.

1 Introduction

Fuzzy integrals are often used as aggregation operators due to their versatility. Among them, we can underline Sugeno [22] and Choquet [3] integrals as they have been used in several applications. See *e.g.*, [10], or [25] for details on such integrals. Fuzzy integrals combine (integrate) a set of numerical values (a function) using fuzzy measures (or capacities) to represent previous information about the data suppliers.

As fuzzy measures are set functions defined on the set of information sources (or data suppliers), their definition requires $2^{|X|}$ parameters. Here, X denotes the set of information sources and $|X|$ is the cardinality of such set. In real applications, the definition becomes unfeasible when the number of sources is of moderate or medium size.

At present, several families of restricted fuzzy measures have been defined to ease the construction of applications. Namely, restricted fuzzy measures are those measures that require less than $2^{|X|}$ parameters because they are constrained to satisfy some additional properties than the ones satisfied by *general* or *unrestricted* ones. Examples of such measures include Sugeno λ-measures [22], $\perp$-decomposable measures (see *e.g.* [10] for details), k-additive fuzzy measures [8], p-symmetric [16], or hierarchically decomposable ones [24].

In this paper we review some recent results on distorted probabilities and m-dimensional distorted probabilities. Distorted probabilities, that were suggested in experimental psychology [4], are measures that can be represented in terms of a probability distribution on X and a non-decreasing function (a distortion function). See *e.g.* [11, 12] or [1] for examples and applications. m-dimensional distorted probabilities is a concept similar to the vector measure game defined in [1]. We include in this paper a new result on m-dimensional probabilities.

This paper has the following structure. First, we give in Section 2 basic definitions that are needed in the rest of the paper. Then, it follows Section 3 that reviews, among other results, a characterization of distorted probabilities. Then, in Section 4 the generalization of distorted probabilities to m-dimensional probabilities is given. Section 5 concludes the paper.

2 Preliminaries

This section is devoted to a review of fuzzy measures. For the sake of simplicity, we will consider X to be a finite set and, in particular, that $X := \{1, 2, \cdots, n\}$.

2.1 Definitions

Definition 2.1. *A function μ on $(X, 2^X)$ is a fuzzy measure if it satisfies the following axioms:*

(i) $\mu(\emptyset) = 0$, $\mu(X) = 1$ *(boundary conditions)*

(ii) $A \subset B$ *implies* $\mu(A) \leq \mu(B)$ *(monotonicity)*

In order to distinguish measures satisfying (i) and (ii) with others that also satisfy some additional constraints (*e.g.* additivity $\mu(A \cup B) = \mu(A) + \mu(B)$ when $A \cap B = \emptyset$), we use the terms *unconstrained* fuzzy measures for the former ones and *constrained* fuzzy measures for the others.

Definition 2.2. *Let f be a real valued function on X and let P be a probability measure on $(X, 2^X)$. We say that f and P represent a fuzzy measure μ on $(X, 2^X)$ if and only if $\mu(A) = f(P(A))$ for all $A \in 2^X$.*

In the case that μ is represented by f and P we will say that f is a distortion function.

Definition 2.3. *Let f be a real valued function on X. We say that f is strictly increasing with respect to a probability measure P if and only if $P(A) < P(B)$ implies $f(P(A)) < f(P(B))$.*

Definition 2.4. *Let μ be a fuzzy measure on $(X, 2^X)$. We say that μ is a distorted probability if it is represented by a probability distribution P on $(X, 2^X)$ and a function f that is strictly increasing with respect to a probability P.*

Remark: Since we suppose that X is a finite set, a strictly increasing function f with respect to P can be regarded as a strictly increasing function on $[0, 1]$ if there is no restriction on the function f. Points except $\{P(A)|A \in 2^X\}$ in $[0, 1]$ are not essential in this paper.

2.2 m-symmetric fuzzy measures

We review now a particular family of fuzzy measures that were proposed by Miranda et al. in [16].

Definition 2.5. *[16, 17] Given a fuzzy measure μ, we say that μ is at most m-symmetric fuzzy measure if and only if there exists a partition of the universal set $\{X_1, \ldots, X_m\}$, with $X_1, \ldots, X_m \neq \emptyset$ such that $X_1, \ldots X_m$ are sets of indifference.*

Definition 2.6. *[16, 17] Given two partitions $\{X_1, \ldots, X_p\}$ and $\{Y_1, \ldots, Y_r\}$ of a referential set X, we say that $\{X_1, \ldots, X_p\}$ is coarser than $\{Y_1, \ldots, Y_r\}$ if the following holds:*

$$\forall X_i \exists Y_j \text{ such that } Y_j \subset X_i$$

Definition 2.7. *[16, 17] Given a fuzzy measure μ, we say that μ is a m-symmetric measure if and only if the coarsest partition of the universal set in sets of indifference is $\{X_1, \ldots, X_m\}$, with $X_i \neq \emptyset$ for all $i \in \{1, \ldots, m\}$.*

3 Distorted probabilities

This section details some of our results on distorted probabilities. A characterization of such measures is included. We start with a result that show that all fuzzy measures can be characterized in terms of a probability distribution and a distortion function, or in terms of a probability distribution and two strictly increasing functions.

Defining the probability measure P on $(X, 2^X)$ such that for all $k \in X$ by

$$P(\{k\}) := \frac{2^{k-1}}{2^n - 1}.$$

we have the next theorem.

Theorem 3.1. *[19] For every fuzzy measure μ on $(X, 2^X)$, there exists a polynomial f and probability P on $(X, 2^X)$ such that $\mu = f \circ P$*

Defining b_k by $b_k := a_k \vee 0$ and c_k by $c_k := -(a_k \wedge 0)$ where the a_k are the coefficients of the polynomial f, we can define

$$f^+(x) := \sum_k b_k x^{2^n - k}$$

and

$$f^-(x) := \sum_k c_k x^{2^n - k}.$$

As f^+ and f^- are strictly increasing, the next corollary holds:

Corollary 3.2. *[19] For every fuzzy measure μ on $(X, 2^X)$, there exists strictly increasing polynomials f^+, f^- and a probability P on $(X, 2^X)$ such that*

$$\mu = f^+ \circ P - f^- \circ P. \quad (1)$$

Therefore, any fuzzy measure can be expressed as the difference of two distorted probabilities.

It is important to underline that neither the difference of two distorted probabilities is a fuzzy measure, nor any distortion function applied to a probability leads to a fuzzy measure. Only non-decreasing distortion functions are assured to lead to a fuzzy measure.

Definition 3.3. *[19] Let μ be a fuzzy measure on $(X, 2^X)$. If $\mu(A) < \mu(B) \Leftrightarrow \mu(A \cup C) < \mu(B \cup C)$ for every $A \cap C = \emptyset$, $B \cap C = \emptyset$ $A, B, C \in 2^X$, we say that μ is a pre-distorted probability.*

Proposition 3.4. *[19] Suppose that a function f and a probability P represent a fuzzy measure μ.*

If f is strictly increasing with respect to P, then $\mu(A) < \mu(B) \Leftrightarrow \mu(A \cup C) < \mu(B \cup C)$ for every $A \cap C = \emptyset, B \cap C = \emptyset$ $A, B, C \in 2^X$. In other words, distorted probabilities are pre-distorted probabilities.

If f is non-decreasing with respect to P, then $\mu(A) < \mu(B)$ implies $\mu(A \cup C) \leq \mu(B \cup C)$ for every $A \cap C = \emptyset$, $B \cap C = \emptyset$ $A, B, C \in 2^X$.

Now, we turn intro the characterization of distorted and pre-distorted probabilities. First, we define a condition that plays a central role in such characterization.

Definition 3.5. *[19] Let μ be a fuzzy measure on $(X, 2^X)$, we say that μ satisfies* condition A *when for all $A_i, B_i \in 2^X$,*

(i) $\sum_{i=1}^n 1_{A_i} = \sum_{i=1}^n 1_{B_i}$

(ii) *$\mu(A_i) \leq \mu(B_i)$ for $i = 2, 3, \ldots, n$ implies $\mu(A_1) \geq \mu(B_1)$*

Now, we give a characterization of distorted probabilities in terms of such condition A.

Theorem 3.6. *[19] Let μ be a fuzzy measure on $(X, 2^X)$, then μ is a distorted probability if and only if Condition A holds.*

In [2], Chateauneuf obtains results similar to the ones in Theorem 3.6. He presents necessary and sufficient conditions for the existence of a non-decreasing distortion function. Here, instead, we consider a strictly increasing polynomial f. Results by Chateauneuf are based on previous results by [6].

Theorem 3.6 and Proposition 3.4 imply the following proposition.

Proposition 3.7. *[19] Let μ be a fuzzy measure on $(X, 2^X)$ satisfying condition A, then μ is a pre-distorted probability.*

The reversal of Proposition 3.7 is not true. Not all pre-distorted probabilities are distorted probabilities. The following example illustrates this situation.

Example 1. *[19] Let $X := \{1, 2, 3\}$, and let μ be the fuzzy measure on 2^X defined as:*

$$\mu(\{1\}) := \frac{1}{7},\ \mu(\{2\}) := 0, \qquad \mu(\{3\}) := \frac{2}{7},$$
$$\mu(\{1,2\}) := \frac{3}{7},\ \mu(\{2,3\}) := \frac{4}{7}, \qquad \mu(\{1,3\}) := 1.$$

Then, this measure is pre-distorted probability because:

$$\mu(\{2\}) < \mu(\{1\}), \mu(\{2,3\}) < \mu(\{1,3\})$$
$$\mu(\{2\}) < \mu(\{3\}), \mu(\{1,2\}) < \mu(\{1,3\})$$
$$\mu(\{1\}) < \mu(\{3\}), \mu(\{1,2\}) < \mu(\{2,3\}).$$

However, this measure does not satisfy condition A. To illustrate this, let us consider the following sets: $A_1 := \{1\}, A_2 := \{2\}, A_3 := \{3\}, A_4 := X$ and $B_1 := \{1,2\}, B_2 := \emptyset, B_3 := \{2,3\}, B_4 := \{1,3\}$. Then, it holds that

$$1_{A_1} + 1_{A_2} + 1_{A_3} + 1_{A_4} = 1_{B_1} + 1_{B_2} + 1_{B_3} + 1_{B_4}$$

and

$$\mu(A_2) \leq \mu(B_2), \mu(A_3) \leq \mu(B_3), \mu(A_4) \leq \mu(B_4)$$

However,

$$\mu(A_1) < \mu(B_1).$$

This example (considered in conjunction with Theorem 3.6) shows that not all pre-distorted probabilities are distorted probabilities. In particular, it shows that there is no suitable strictly increasing distortion function f to represent μ. Nevertheless, the fuzzy measure μ could be indeed represented with a non-decreasing function f. This is shown in the next example.

Example 2. *[19] Let us consider X and the measure μ defined in Example 1, then, there exists probability distributions P and non-decreasing functions f such that $\mu = f \circ P$. Table 1 gives one of such probability distributions together with the fuzzy measure μ.*

set	$\emptyset$	{2}	{1}	{3}
P	0	1/5.5	2/5.5	2.5/5.5
$\mu = f \circ P$	0	0	1/7	2/7
{1,2}	{2,3}	{1,3}	{1, 2, 3}	
3/5.5	3.5/5.5	4.5/5.5	5.5/5.5	
3/7	4/7	7/7	7/7	

Table 1: Probability and non-decreasing function f corresponding to Example 1

This example shows that with a non-decreasing function f, the fuzzy measure μ of Example 1 can be represented as $\mu = f \circ P$. Nevertheless, this function f is no longer a polynomial.

We now turn into the study of distorted probabilities. We will base our study on a condition A' that is similar to condition A. This condition will be based on an equivalence relation $\sim$ on 2^X and a quotient set.

Definition 3.8. *[19] Let $\sim$ on 2^X be the equivalence relation defined by:*

$$A \sim B \Leftrightarrow \mu(A) = \mu(B)$$

for $A, B \in 2^X$.

Now, given $\sim$, let us consider the quotient set $2^X / \sim$ (*i.e.*, if $[A] \in 2^X / \sim$, $B \in [A]$ implies $A \sim B$) and, then, let $\mathcal{B}_\mu$ denote its representatives ($\emptyset$ and X are considered in $\mathcal{B}_\mu$). Naturally, $A, B \in \mathcal{B}_\mu$ implies either $\mu(A) < \mu(B)$ or $\mu(B) < \mu(A)$. Let L be the real linear vector space generated by the set of characteristic functions $1_A : A \in 2^X$ and let $\mathcal{X}_\mu$ of L be defined by

$$\mathcal{X}_\mu := \{1_A - 1_B | A, B \in \mathcal{B}_\mu\}.$$

Definition 3.9. *[19] The function $f : [0, 1] \to [0, 1]$ is said to be strictly increasing with respect to $\mathcal{B}_\mu$ if and only if $P(A) < P(B)$ implies $f(P(A)) < f(P(B))$ for $A, B \in \mathcal{B}_\mu$.*

Finally, we define condition A'. This condition is analogous to condition A (Definition 3.5) but restricted to sets in $\mathcal{B} \subset 2^X$.

Definition 3.10. *[19] Let μ be a fuzzy measure on $(X, 2^X)$, and $\mathcal{B}$ be a subset of 2^X, then we say that μ satisfies* condition A' *when for all $A_i, B_i \in \mathcal{B}$,*

(i) $\sum_{i=1}^{n} 1_{A_i} = \sum_{i=1}^{n} 1_{B_i}$

(ii) $\mu(A_i) \leq \mu(B_i)$ *for* $i = 2, 3, \ldots, n$ *implies* $\mu(A_1) \geq \mu(B_1)$

Suppose that $A, B \in \mathcal{B}_\mu$ and $\mu(A) \leq \mu(B)$. It follows from the definition of $\mathcal{B}_\mu$ that the equality occur if and only if $A = B$.

Theorem 3.11. *[19] Let μ be a fuzzy measure on $(X, 2^X)$. There exists a probability P on $(X, 2^X)$ and a polynomial f which is strictly increasing with respect to $\mathcal{B}_\mu$ such that*

$$\mu = f \circ P$$

if and only if condition A' holds for $\mathcal{B}_\mu$.

Corollary 3.12. *[19] Let μ be a fuzzy measure on $(X, 2^X)$. If μ is represented by a probability on $(X, 2^X)$ and a non-decreasing function f, then condition A' holds.*

The following examples illustrates previous results.

Example 3. *[19] Let $X := \{1,2,3\}$, $\mu(\emptyset) = \mu(\{1\}) = \mu(\{2\}) = 0$, $\mu(\{3\}) = 0.5$ $\mu(\{1,2\}) = 0.7$, $\mu(\{1,3\}) = \mu(\{2,3\}) = \mu(X) = 1$. Then $\mathcal{B}_\mu = \{\emptyset, \{3\}, \{1,2\}, X\}$. Since we have*

$$1_{\{1,2\}} + 1_{\{3\}} = 1_\emptyset + 1_X,$$

$\mu(\{1,2\}) < \mu(X)$ and $\mu(\{3\}) > \mu(\emptyset)$. Therefore there exist a probability P and a strictly increasing polynomial with respect to $\mathcal{B}_\mu$ such that $\mu = f \circ P$

Example 4. *[19] Let $X := \{1,2,3,4\}$, $\mu(\emptyset) = \mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = \mu(\{4\}) = 0$, $\mu(\{1,2\}) = 0.3$, $\mu(\{2,3\}) = 0.2$, $\mu(\{1,4\}) = 0.1$, $\mu(\{3,4\}) = 0.4$, and $\mu(A) = 1$ otherwise. This fuzzy measure cannot be representable in terms of a distortion function. There is no strictly increasing polynomial w.r.t. $\mathcal{B}_\mu$ and a probability P such that $\mu = f \circ P$. This is so because we have*

$$\mathcal{B}_\mu := \{\emptyset, \{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, X\},$$

such that

$$1_{\{1,2\}} + 1_{\{3,4\}} = 1_{\{1,4\}} + 1_{\{2,3\}},$$

with $\mu(\{1,2\}) > \mu(\{2,3\})$ and $\mu(\{3,4\}) > \mu(\{1,4\})$.

Definition 3.13. *[19] Let $a_1, a_2, \ldots, a_m \in R^n$. The convex cone $[a_1, a_2, \ldots, a_m]$ generated by $\{a_1, a_2, \ldots, a_m\}$ is defined by*

$$[a_1, a_2, \ldots, a_m] := \{\sum_{i=1}^{m} \lambda_i a_i | \lambda_i \geq 0\}.$$

Theorem 3.14. *[19] Let μ be a fuzzy measure on $(X, 2^X)$ with $\mathcal{B}_\mu = 2^X$. If there exist linearly independent vectors $a_1, a_2, \ldots, a_m \in R^n$ such that $\mathcal{X}_\mu \subset [a_1, a_2, \ldots, a_m] \setminus 0$, then μ is a distorted probability.*

Corollary 3.15. *[19] Let μ be a pre-distorted probability on $(X, 2^X)$.*

If $|X| = \{1,2,3,4\}$ and $\mathcal{B}_\mu = 2^X$, then μ is a distorted probability.

4 m-dimensional distorted probabilities

In this section we review our definition of m-dimensional distorted probabilities. The introduction of such measures is motivated by the fact that distorted probabilities are only a small proportion of all possible fuzzy measures. m-dimensional distorted probabilities are defined in a way that with increasing values of m they cover the whole range between distorted probabilities and general (unrestricted) fuzzy measures.

4.1 Definitionand basic properties

Definition 4.1. *[19] Let $\{X_1, X_2, \cdots, X_m\}$ be a partition of X (i.e., $\cup X_i = X$ and for all X_i and X_j it holds $X_i \cap X_j = \emptyset$), then we say that μ is at most m-dimensional pre-distorted probability if for all X_i it holds:*

$$\mu(A) \geq \mu(B) \Leftrightarrow \mu(A \cup C) \geq \mu(B \cup C) \quad (2)$$

for all $A, B, C \subset X_i$ such that $C \cap A = \emptyset$ and $C \cap B = \emptyset$. We say that at most m-dimensional pre-distorted probability μ is m-dimensional pre-distorted probability if μ is not at most $(m-1)$-dimensional.

From the above definition for pre-distorted probabilities, it is clear that:

Proposition 4.2. *[19] All fuzzy measures are at most $|X|$-dimensional pre-distorted probabilities.*

Definition 4.3. *[19] Let $\{X_1, X_2, \cdots, X_m\}$ be a partition of X (i.e., $\cup X_i = X$ and for all X_i and X_j it holds $X_i \cap X_j = \emptyset$), then we say that μ is at most m-dimensional distorted probability if there exists a function f on R^m and probability P_i on $(X_i, 2^{X_i})$ such that*

$$\mu(A) = f(P_1(A \cap X_1), P_2(A \cap X_2), \cdots, P_3(A \cap X_m)) \quad (3)$$

where f is strictly increasing with respect to the i-th axis for all $i = 1, 2, \ldots, m$. We say that at most m-dimensional distorted probability μ is m-dimensional pre-distorted probability if μ is not at most $(m-1)$-dimensional.

Definition 4.3 implies the next proposition.

Proposition 4.4. *[19] All fuzzy measures are at most $|X|$-dimensional distorted probabilities.*

Also, the following holds:

Proposition 4.5. *Let $\mathcal{M}_k$ be the set of all fuzzy measures that are k-dimensional distorted probabilities, then $\mathcal{M}_{k-1} \subset \mathcal{M}_k$ for all $k = 2, 3, \ldots, |X|$.*

Corollary 4.6. *Given a fuzzy measure μ, there exists a $k = 1, 2, \ldots, |X|$ such that $\mu \in \mathcal{M}_k$ and $\mu \notin \mathcal{M}_{k-1}$.*

Therefore, the proposed family of fuzzy measures permits to cover the whole set of fuzzy measures.

4.2 On m-dimensional OWA and m-dimensional WOWA

The definition of m-dimensional distorted probability permits to define a generalization of the WOWA operator. This operator, introduced in [23], was proven to be equivalent to a Choquet integral with respect to a distorted probability. Based on such interpretation of the WOWA operator, we define an m-dimensional version as a Choquet integral with respect to a m-dimensional distorted probability.

Definition 4.7. *[20] The m-dimensional WOWA is defined as the Choquet integral with respect to a m-dimensional distorted probability.*

This result extends another result that defines in a similar way the m-dimensional OWA in terms of m-symmetric fuzzy measures.

Definition 4.8. *[20] The m-dimensional OWA is defined as the Choquet integral with respect to a m-symmetric fuzzy measure.*

4.3 Some relationships with m-symmetric fuzzy measures

Let us now consider some results that establish the connections between m-symmetric fuzzy measures and m-dimensional distorted probabilities.

Proposition 4.9. *[20] Let μ be a m-symmetric fuzzy measure with respect to the partition $\{X_1, \dots, X_m\}$, then, μ is a $m-$dimensional distorted probability.*

Corollary 4.10. *[20] m-dimensional OWA is a particular case of m-dimensional WOWA. In other words, a Choquet integral with respect to a m-symmetric fuzzy measure is a particular case of a Choquet integral with respect to a m-dimensional distorted probability.*

As shown in Proposition 4.9, one special class of at most $m-$dimensional probability are one special class of $m-$symmetric fuzzy measures. We present another special class of m-dimensional distorted probability.

Definition 4.11. *[20] Let μ be a $m-$dimensional distorted probability.*

We say that μ is a type 2 m-symmetric fuzzy measure if there exists functions $f_1, \dots, f_m$ and g such that

$$\mu(A) = g(f_1 \circ P_1(A \cap X_1), \dots, f_m \circ P_m(A \cap X_m)) \quad (4)$$

where g is a symmetric function on R^m.

We present now an example of type 2 m-symmetric fuzzy measure.

Example 5. *Let us consider an entrance examination consisting on two parts: Mathematics and English. Each part is divided in several exercises. On the one hand, the grade on mathematics is defined by the grade on three exercices: algebra (x_1), geometry (x_2) and probability (x_3). On the other hand, the rate of English is computed in terms of two exercises: grammar (y_1) and reading (y_2).*

*In this case, it seems natural that there is some interaction within mathematical exercises (*i.e., *within $\{x_1, x_2, x_3\}$) and within English exercises (*i.e., *within $\{y_1, y_2\}$) but not among an English and a Mathematics exercise. I.e., the scores of Mathematics and English are treated equally. This situation can be modeled in terms of a type 2 2-symmetric fuzzy measure. Symmetry of g precisely means that Mathematics and English are evaluated equally, and that there is no interaction between Mathematics and English. For the interaction among exercises in Mathematics (or English) the distortion function f_1 (or f_2) can be used on the probabilities (or importances) of the corresponding exercises.*

Type 2 m-symmetric fuzzy measures are also a generalization of 1-symmetric fuzzy measures.

Proposition 4.12. *[20] A type 2 $n-$symmetric fuzzy measure μ is a $1-$symmetric fuzzy measure if $f_i(x) = x$ for all $1 \leq i \leq n$.*

Theorem 4.13. *[20] Let $\{X_1, \dots, X_m\}$ be a partition of X and μ_i $i = 1, \dots m$ be distorted probabilities. Then there exists a type 2 m-symmetric fuzzy measure μ on $(X, 2^X)$ such that*

$$\sum_{i=1}^{m}((C)\int f df_i(P_i)) = (C)\int f d\mu \quad (5)$$

for all measurable function f.

4.4 Representation of m-dimensional distorted probabilities

Let f be a m-dimensional continuous function on $[0,1]^m$, that is, $f : [0,1]^m \to [0,1]$. Kolmogorov [14] showed that there exist one variable real valued continuous function χ_i, $\phi_{i,j}$: $R \to R$, $(i = 1, 2, \dots, 2n+1)$, $(j = 1, \dots, n)$ such that

$$f(x_1, x_2, \dots, x_n) := \sum_{i=1}^{2n+1} \chi_i(\sum_{j=1}^{n} \phi_{i,j}(x_j)),$$

where every $\phi_{i,j}(x_j)$ is monotone and does not depend on f. The Kolmogolov theorem is well known as the complete solution to the Hilbert's thirteenth problem. It follows from construction method in Kolmogorov theorem that $\phi_{i,j}(0) = 0$. Let $M_{i,j} := max_{x \in [0,1]} \phi_{i,j}(x)$ and $f_{i,j} := \phi_{i,j}/M_{i,j}$. Since $f_{i,j}(0) = 0$ and $f_{i,j}(1) = 1$,a set function $f_{i,j}(P_j(A)) := \mu_{i,j}(A)$ is a fuzzy measure. Therefore we have the next proposition.

Proposition 4.14. *Let μ be a m-dimensional distorted probability. There exist one variable real valued continuous functions $\chi_i : R \to R, (i = 1, 2, \dots, 2m+1)$, non-negative real numbers $a_{i,j}$ 1 dimensional distorted probabilities $\mu_{i,j}$ $(i = 1, \dots, 2m+1)$, $(j = 1, \dots m)$ such that*

$$\mu(A) = \sum_{i=1}^{2m+1} \chi_i(\sum_{j=1}^{m} a_{i,j}\mu_{i,j}(A))$$

for $A \in 2^X$.

The proposition above says that a m-dimensional fuzzy measure can caluculate in the following 3-steps;

1: for every i, j: 1 dimensional distorted probabilities $a_{i,j}\mu_{i,j}(A)$,

2: for every i,a special form of m-dimensional distorted probability: $\nu_i(A) := \sum_{j=1}^{m} a_{i,j}\mu_{i,j}(A)$,

3: $\mu(A) := \sum_{i=1}^{2m+1} \chi_i(\nu_i(A))$.

5 Conclusions

In this paper we have reviewed our recent results on distorted probabilities and on m-dimensional distorted probabilities. We have given some representation results as well as the connections between such measures and m-symmetric ones. Additionally, we have given some new results on the calculation of the m-dimensional distorted probabilities.

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