Relationships among continuity conditions and null-additivity conditions in non-additive measure theory

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***Abstract*— The paper shows all the implication relationships among six continuity conditions and two null-additivity conditions. The six continuity conditions are continuity from above, order continuity, strong order continuity, exhaustivity, continuity from below, and null-continuity, and the two null-additivity conditions are null-additivity and weak null-additivity.**

I. Introduction

So far, non-additive measure theory has been constructed along lines of the classical measure theory [1, 9]; here a non-additive measure means a non-negative monotone set function. But in general, theorems in the classical measure theory do not hold in non-additive measure theory without additional conditions. So several conditions with respect to non-additive measures have been given in order that such theorems hold [2, 3, 4, 8, 9]. For example, Egoroff's theorem, which is one of the most important convergence theorems in the classical measure theory, does not hold in non-additive measure theory, but it holds under continuity from above and below, that is, the continuity from above and below is a sufficient condition for the theorem [5]. In the same way, sufficient conditions for other theorems have been given [2, 3, 4, 8, 9], but the relationships among these conditions hardly have been studied so far. So it is very important to clarify the relationships among them. In this paper, we investigate eight well-known conditions with respect to continuity and null additivity, and show all the implication relationships among them.

Figure 1 summarizes the obtained implication relationships. In this diagram, a directed path from A to B means that condition A implies condition B, and the absence of such a directed path means that A does not imply B; for example, continuity from above implies strong order continuity, null-continuity does not imply continuity from below, and continuity from below does not imply null-additivity.

This paper is organized as follows. In Section 2, we give the definitions of a non-additive measure and eight conditions discussed in this paper. In Section 3, the relationships of all the arrows in Fig. 1 are stated. In Section 4, by giving several examples of non-additive measures, we show that there are no arrows other than those in Fig.1. In Section 5, we state the conclusion of this paper, and give a subject of future research.

II. Definitions

Throughout the paper, we assume that $(X, \mathfrak{F})$ is a measurable space and $\mathbb{N}$ denotes the set of positive integers.

Definition 1 A *non-additive measure* on $\mathfrak{F}$ is a set function $\mu : \mathfrak{F} \to [0, \infty]$ satisfying the following two conditions:

(i) $\mu(\emptyset) = 0$,

(ii) $A, B \in \mathfrak{F}, A \subset B \Rightarrow \mu(A) \leq \mu(B)$.

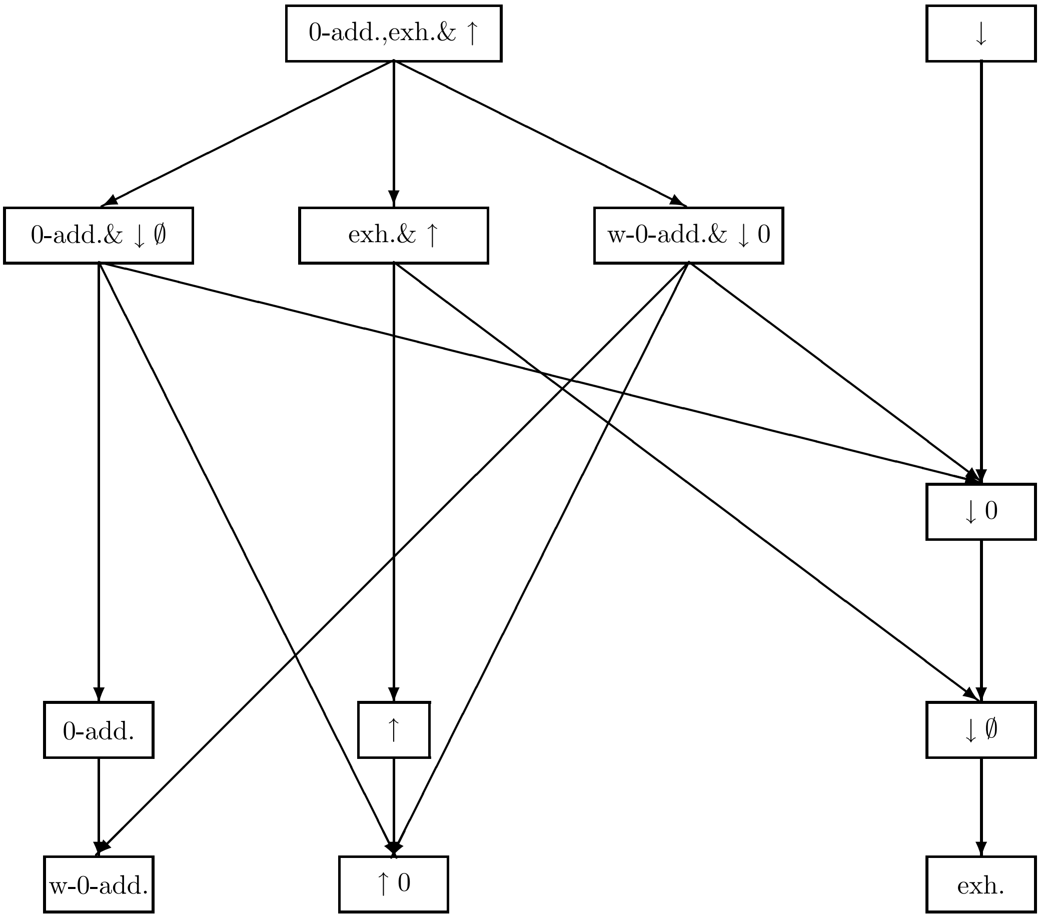
In this paper, we assume that μ is a non-additive measure on $\mathfrak{F}$ and that all subsets are $\mathfrak{F}$-measurable.

We investigate eight conditions in this paper. First, we give six conditions concerning continuity of non-additive measures.

Definition 2 (i) μ is said to be *continuous from above* if $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$ for every decreasing sequence $\{A_n\}$.

(ii) μ is said to be *order continuous* if it is continuous at the empty set, that is, it holds that $\lim_{n\to\infty} \mu(A_n) = 0$ whenever $A_n \downarrow \emptyset$ [2].

(iii) μ is said to be *strongly order continuous* if it is continuous at measurable sets of measure zero,



0-add. : null-additivity
↓ : continuity from above
↓ ∅ : order continuity
↑ : continuity from below
w-0-add. : weak null-additivity
↓ 0 : strong order continuity
exh. : exhaustivity
↑ 0 : null-continuity

Figure 1: Implication relationships

that is, it holds that $\lim_{n\to\infty} \mu(A_n) = 0$ whenever $A_n \downarrow A$ and $\mu(A) = 0$ [4].

(iv) μ is said to be *exhaustive* if $\lim_{n\to\infty} \mu(A_n) = 0$ for every disjoint sequence $\{A_n\}$ [9].

(v) μ is said to be *continuous from below* if $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$ for every increasing sequence $\{A_n\}$.

(vi) μ is said to be *null-continuous* if $\mu(\bigcup_{n=1}^{\infty} A_n) = 0$ for every increasing sequence $\{A_n\}$ such that $\mu(A_n) = 0$ for any n [6].

Secondly, we give two conditions concerning null-additivity of non-additive measures.

Definition 3 (vii) μ is said to be *null-additive* if $\mu(A \cup B) = \mu(A)$ for every set B such that $\mu(B) = 0$ [8].

(viii) μ is said to be *weakly null-additive* if $\mu(A \cup B) = 0$ for every pair of sets A and B such that $\mu(A) = \mu(B) = 0$ [9].

III. Implication Relationships

In this section, we state the relationship of all the arrows in Fig.1.

First, we give the implication relationships from one condition to another. The following proposition have already been given in [2, 4, 6].

Proposition 1 (i) *If μ is continuous from above, then it is strongly order continuous.*

(ii) *If μ is strongly order continuous, then it is order continuous.*

(iii) *If μ is order continuous, then it is exhaustive.*

(iv) *If μ is null-additive, then it is weakly null-additive.*

(v) *If μ is continuous from below, then it is null-continuous.*

Note that the converses of (ii)–(v) in Proposition 1 do not hold [2, 4, 6], and the converse of (i) does not hold as shown in Example 1 of Section IV.

Except as stated in Proposition 1, conditions (i)–(viii) of Definitions 2 and 3 are pairwise independent [6].

Next, we give the implication relationships from more than one conditions to another.

Proposition 2 [4] *If μ is null-additive and order continuous, then it is strongly order continuous.*

Proposition 3 [3] *If μ is exhaustive and continuous from below, then it is order continuous.*

By these two propositions, we have the following corollary.

Corollary 1 If μ is null-additive, exhaustive and continuous from below, then it is strongly order continuous.

Proposition 4 [7] *If μ is null-additive and order continuous, then it is null-continuous.*

The following is the main proposition of this paper.

Proposition 5 If μ is weakly null-additive and strongly order continuous, then it is null-continuous.

Proof: Assume that μ is weakly null-additive and strongly order continuous. Let $N_n \uparrow N$ and $\mu(N_n) = 0$ for all $n \geq 1$. We define sequences of sets $\{A_i\}$ and $\{B_i\}$ recursively by

$$\begin{aligned} A_0 &= \emptyset, \\ B_i &= N_{n_{(2i-1)}} \setminus A_{i-1} \quad \text{for } i \geq 1, \\ A_i &= N_{n_{2i}} \setminus B_i \quad \text{for } i \geq 1, \end{aligned}$$

where $\{n_k\}$ is the strictly increasing sequence defined below. Note that $\mu(A_i) = 0$ and $\mu(B_i) = 0$ for all i since $A_0 = \emptyset$, $B_i \subset N_{n_{(2i-1)}}$ and $A_i \subset N_{n_{2i}}$. Let $n_0 = 0$. We define $n_{(2i-1)}$ for $i \geq 1$ as follows. Since

$$(A_{i-1} \cup (N \setminus N_n)) \downarrow A_{i-1} \text{ as } n \to \infty,$$

and since $\mu(A_{i-1}) = 0$, by strongly order continuity we can choose $n_{(2i-1)}$ so that $n_{(2i-1)} > n_{(2i-2)}$ and

$$\mu\left(A_{i-1} \cup \left(N \setminus N_{n_{(2i-1)}}\right)\right) < \frac{1}{i}.$$

Similarly, for $i \geq 1$ we can choose n_{2i} so that $n_{2i} > n_{2i-1}$ and

$$\mu\left(B_i \cup \left(N \setminus N_{n_{2i}}\right)\right) < \frac{1}{i}.$$

Now let $A = \bigcup_{i=1}^{\infty} A_i$ and $B = \bigcup_{i=1}^{\infty} B_i$. Then for every i, since $A \subset A_{i-1} \cup \left(N \setminus N_{n_{(2i-1)}}\right)$, it follows that

$$\mu(A) \leq \mu\left(A_{i-1} \cup \left(N \setminus N_{n_{(2i-1)}}\right)\right) < \frac{1}{i}.$$

Hence $\mu(A) = 0$. Similarly we have $\mu(B) = 0$. Since $A \cup B = N$, weakly null-additivity implies that

$$\mu(N) = \mu(A \cup B) = 0.$$

Therefore μ is null-continuous. ■

IV. Examples

In this section, we give several examples of non-additive measures. These examples show that there are no implication relationships other than shown in the previous section.

We summarize the following examples as Table 1. In this table, a symbol "◯" in the cell at the row of "Ex.A" and the column of a condition "C" means that Example A satisfies the condition C, and a symbol "×" means that Example A does not satisfy the condition C; for instance, Example 1 satisfies null-additivity and not continuity from above.

Example 1 Let $X = \mathbb{N}$, $m(A) = \sum_{n \in A} 1/2^n$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} m(A) & \text{if } m(A) \leq 1/2, \\ 1 & \text{if } m(A) > 1/2. \end{cases}$$

Then obviously μ satisfies the following conditions:
(1) strong order continuity,
(2) continuity from below,
(3) null-additivity.
By (1), μ is order continuous and exhaustive. By (3), it is weakly null-additive. But it is not continuous from above.

Table 1: Summary of the examples

Example.	0-add.	w-0-add.	$\downarrow$	$\downarrow 0$	$\downarrow \emptyset$	exh.	$\uparrow$	$\uparrow 0$
Ex.1	○	○	×	○	○	○	○	○
Ex.2	×	×	○	○	○	○	○	○
Ex.3	×	○	○	○	○	○	○	○
Ex.4	○	○	×	×	×	×	○	○
Ex.5	×	○	×	×	○	○	○	○
Ex.6	×	○	×	×	○	○	×	×
Ex.7	○	○	×	×	×	○	×	○
Ex.8	○	○	×	×	×	○	×	×
Ex.9	○	○	○	○	○	○	×	○

0-add. : null-additivity
$\downarrow$: continuity from above
$\downarrow \emptyset$: order continuity
$\uparrow$: continuity from below
w-0-add. : weak null-additivity
$\downarrow 0$: strong order continuity
exh. : exhaustivity
$\uparrow 0$: null-continuity

Example 2 Let $X = \{0, 1\}$, μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 0 & \text{if} \quad A = \emptyset, \{0\} \text{ or } \{1\}, \\ 1 & \text{if} \quad A = X. \end{cases}$$

Then obviously μ satisfies the following conditions:
(1) continuity from above,
(2) continuity from below.
By (1), μ is strongly order continuous, order continuous and exhaustive. By (2), it is null-continuous. But it is not weakly null-additive; hence is not null-additive.

Example 3 Let $X = \{0, 1\}$, μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 0 & \text{if} \quad A = \emptyset \text{ or } \{0\}, \\ 1 & \text{if} \quad A = \{1\}, \\ 2 & \text{if} \quad A = X. \end{cases}$$

Then clearly μ satisfies the following conditions:
(1) continuity from above,
(2) continuity from below,
(3) weak null-additivity.
By (1), μ is strongly order continuous, order continuous and exhaustive. By (2), it is null-continuous. But it is not null-additive.

Example 4 Let $X = \mathbb{N}$, μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 0 & \text{if} \quad A = \emptyset, \\ 1 & \text{if} \quad A \neq \emptyset. \end{cases}$$

Then μ satisfies the following conditions:
(1) continuity from below,
(2) null-additivity.
By (1), μ is null-continuous. By (2), it is weakly null-additive. But it is not exhaustive; hence it is not continuous from above, strongly order continuous and order continuous.

Example 5 Let $X = \mathbb{N}$, $m(A) = \sum_{n \in A} 1/2^n$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 0 & \text{if} \quad A = \{1\}, \\ m(A) & \text{if} \quad A \neq \{1\}. \end{cases}$$

Then μ satisfies the following conditions:
(1) order continuity,
(2) continuity from below,
(3) weak null-additivity.
By (1), μ is exhaustive. By (2), it is null-continuous. But it is neither null-additive nor strongly order continuous; hence it is not continuous from above.

Example 6 Let $X = \mathbb{N}$, $m(A) = \sum_{n \in A} 1/2^n$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 0 & \text{if} \quad A \text{ is finite}, \\ m(A) & \text{if} \quad A \text{ is infinite}. \end{cases}$$

Then μ satisfies the following conditions:
(1) order continuity,
(2) weak null-additivity.
By (1), μ is exhaustive. But it is not strongly order continuous; hence it is not continuous from above.

Furthermore it is not null-continuous; hence it is not continuous from below. Then μ is not null-additive, because order continuity and null-additivity imply null-continuity (Prop. 4).

Example 7 Let $X = \mathbb{N}$, $m(A) = \sum_{n \in A} 1/2^n$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} m(A) & \text{if} \quad A^c \text{ is infinite,} \\ 1 & \text{if} \quad A^c \text{ is finite.} \end{cases}$$

Then μ satisfies the following conditions:
(1) null-additivity,
(2) exhaustivity,
(3) null-continuity.
By (1), μ is weakly null-additive. But it is not order continuous; hence it is not strongly order continuous and continuous from above. Then μ is not continuous from below, because exhaustivity and continuity from below imply order continuity (Prop. 3).

Example 8 Let $X = \mathbb{N}$, $\mathfrak{M} = \{M \subset X \mid |M^c| < \infty\}$, $\mathfrak{A}$ be a ultra filter such that $\mathfrak{M} \subset \mathfrak{A}$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 0 & \text{if} \quad A \notin \mathfrak{A}, \\ 1 & \text{if} \quad A \in \mathfrak{A}. \end{cases}$$

Then μ satisfies the following conditions:
(1) null-additivity,
(2) exhaustivity.
By (1), μ is weakly null-additive. But it is not order continuous; hence it is not strongly order continuous and continuous from above. Furthermore it is not null-continuous; hence it is not continuous from below.

The following example is given in [6].

Example 9 [6] Let $r > 1$, $X = [0, r]$ and λ be a Lebesgue measure on $\mathbb{R}$. Furthermore let μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} \lambda(A) & \text{if} \quad \lambda(A) < 1, \\ 2 & \text{if} \quad \lambda(A) \geq 1. \end{cases}$$

Then μ satisfies the following conditions:
(1) null-additivity,
(2) continuity from above,
(3) null-continuity.
By (1), μ is weakly null-additive. By (2), it is strongly order continuous, order continuous and exhaustive. But it is not continuous from below.

V. Concluding Remarks

We have showed all the implication relationships among the eight conditions with respect to non-additive measures: continuity from above, order continuity, strong order continuity, exhaustivity, continuity from below, null-continuity, null-additivity and weak null-additivity. We can summarize these implication relationships as in Fig.1.

In this paper, we have dealt only with the eight conditions, but there are many other conditions which are given as a sufficient condition in order that a theorem holds. So we are investigating the implication relationships among other conditions such as auto-continuity, property S, and pseudometric generating property.

References

[1] D. Denneberg. Non-additive Measure and Integral. Kluwer, 1997.

[2] I. Dobrakov, On submeasures I, Dissertationes Math. 112 (1974) 1–35.

[3] I. Dobrakov, J. Farkova, On submeasures II, Math. Slovaca, 30 (1980) 65–81.

[4] J. Li, Order continuous of monotone set function and convergence of measurable function sequence, Appl. Math. Comput. 135 (2003) 211–218.

[5] J. Li, On Egoroff theorems on fuzzy measure spaces, Fuzzy Sets and Systems 135 (2003) 367–375.

[6] K. Uchino, T. Murofushi, Relations between mathematical properties of fuzzy measures, Proc. 10th IFSA World Congr., Istanbul, Turkey, (2003) 27–30.

[7] K. Uchino, S. Asahina, T. Murofushi, Dense pack property and null-continuity of fuzzy measure, in preparation.

[8] Z. Wang, The autocontinuity of set function and the fuzzy integral, J. Math. Anal. Appl., 99 (1984) 195–218.

[9] Z. Wang, G.J. Klir, Fuzzy Measure Theory, Plenum, New York, 1992.