Cournot Equilibrium with Possibilistic Information

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***Abstract*— In this paper, the uncertainty of duopoly market is characterized by the possibility distribution, the new Cournot equilibrium, called optimistic Cournot equilibrium and pessimistic equilibrium, are proposed to analyze duopoly market.**

I. Cournot Equilibrium with Symmetrical Possibilistic Information

We image an industry with only firm 1 and firm 2 (duopoly market), each producing and selling a single good. Consumers do not care from which firm they purchase the good. A very concrete and simple case of linear demand is considered as follows:

$$P = a - b(q_1 + q_2)\,, \quad (P > 0) \tag{1}$$

where p is price, q_1 and q_2 are sale quantities from Firm 1 and Firm 2, respectively, $b > 0$ is a constant.
The plausible information on market is characterized by the possibility distribution of a in (1).

Definition 1. The possibility distribution of a, denoted as π_A, is defined by the following continuous function

$$\pi_A : [a_l, a_r] \to [0,1]\,, \tag{2}$$

where a_l and a_r are the lower and upper bounds of a. $\exists a_c \in [a_l, a_u]$ so that $\pi_A(a_c) = 1$, $\pi_A(a_l) = 0$, $\pi_A(a_r) = 0$. π_A increases within $[a_l, a_c]$ and decreases within $[a_c, a_r]$.

Proposition 1. The region of the profit of firm i is $[0, \frac{a_r^2}{4b}]$.

Definition 2. The utility function of firm i (i=1, 2) is defined by the following strictly increasing function,

$$u : [0, \frac{a_r^2}{4b}] \to [0,1] \tag{3}$$

where $u(0) = 0$ and $u(\frac{a_r^2}{4b}) = 1$.

Definition 3. Conjecturing the output level of firm j being q_j and possibility degree of a, the optimistic value of the output level q_i of firm i, denoted as $V_{io}(q_i, q_j)$ is defined as follows:

$$V_{io}(q_i, q_j) = \max_a(\min(\pi_A(a), u(w_i(q_i, q_j, a))))\,, \tag{4}$$

which is called as optimistic criterion.

It can be seen from (4) that a higher evaluation value of q_i is given by optimistic criterion if q_i can produce a higher utility with higher possibility, which is of risk-preference.

Definition 4. Conjecturing the output level of firm j being q_j and possibility degree of a, the pessimistic value of the output level q_i of firm i, denoted as $V_{ip}(q_i, q_j)$ is defined as follows

$$V_{ip}(q_i, q_j) = \min_a(\max(1 - \pi_A(a), u(w_i(q_i, q_j, a))))\,, \tag{5}$$

,which is called as pessimistic criterion.

It can be seen from (5) that a lower evaluation value of q_i is given by pessimistic criterion if q_i can produce a lower utility with higher possibility, which is of risk-aversion.

It is reasonable that firm i chooses its output level according to the following criteria:
(I) Firm i chooses its output level q_i to maximize $V_{io}(q_i, q_j)$. That is,

$$q_i = \arg\max_{q_i} V_{io}(q_i, q_j) = g_{io}(q_j)\,, \tag{6}$$

which is a reaction function of Firm i, called optimistic reaction function.

(II) Firm i chooses its output level q_i to maximize $V_{ip}(q_i, q_j)$. That is,

$$q_i = \arg\max_{q_i} V_{ip}(q_i, q_j) = g_{ip}(q_j)\,, \tag{7}$$

which is a reaction function of Firm i, called pessimistic reaction function.
Definition 5. The solutions of the following equations (8) and (9), denoted as (q_{1o}^*, q_{2o}^*) and (q_{1p}^*, q_{2p}^*), are called as

optimistic Cournot equilibrium and pessimistic Cournot equilibrium, respectively.

$$\begin{cases} q_1 = g_{1o}(q_2) \\ q_2 = g_{2o}(q_1) \end{cases} \tag{8}$$

$$\begin{cases} q_1 = g_{1p}(q_2) \\ q_2 = g_{2p}(q_1) \end{cases} \tag{9}$$

Proposition 2. The optimistic Cournot equilibrium (q_{1o}^*, q_{2o}^*) is the solution of the following equations.

$$\begin{cases} q_1 = \dfrac{\hat{a}_{1o}(q_2) - bq_2}{2b} \\ q_2 = \dfrac{\hat{a}_{2o}(q_1) - bq_1}{2b} \end{cases} \tag{10}$$

where $\hat{a}_{io}(q_j)$ is the horizontal coordinate of the right intersection of $\pi_A(a)$ and $u(w_i(a, q_i^*(q_j), q_j))$, which is called as optimistic focus point of Firm i to represent the most considerable value of a by the optimistic viewpoint.

Corollary 1. $\hat{a}_{1o} = \hat{a}_{2o}$, $q_{1o}^* = q_{2o}^*$

Proposition 3. The pessimistic Cournot equilibrium (q_{1p}^*, q_{2p}^*) is the solution of the following equations.

$$\begin{cases} q_1 = \dfrac{\hat{a}_{1p}(q_2) - bq_2}{2b} \\ q_2 = \dfrac{\hat{a}_{2p}(q_1) - bq_1}{2b} \end{cases}, \tag{11}$$

where $\hat{a}_{ip}(q_j)$ is the horizontal coordinate of the left intersection of $1 - \pi_A(a)$ and $u(w_i(a, q_i^*, q_j))$, which is called as pessimistic focus point of firm i to represent the most considerable value of a from pessimistic viewpoint.

Corollary 2. $\hat{a}_{1p} = \hat{a}_{2p}$, $q_{1p}^* = q_{2p}^*$

Theorem 1. There is one and only one optimistic Cournot equilibrium (q_{1o}^*, q_{2o}^*) and pessimistic Cournot equilibrium (q_{1p}^*, q_{2p}^*) for these two firms.

Theorem 2. The optimistic equilibrium outcomes are large than the pessimistic equilibrium outcomes, that is, $[q_{1o}^*, q_{2o}^*] > [q_{1p}^*, q_{2p}^*]$.

Definition 6. Suppose that there are two possibility distributions, π_A and π_B. If for any arbitrary x $\pi_A(x) \geq \pi_B(x)$ holds, then π_B is said to be more informed than π_A, which is denoted as $\pi_A \geq \pi_B$.

Theorem 3. Suppose that $(q_{1o}^{a*}, q_{2o}^{a*})$ and $(q_{1o}^{b*}, q_{2o}^{b*})$ are the optimistic Cournot equilibriums based on possibility distributions π_A and π_B, respectively, $(q_{1p}^{a*}, q_{2p}^{a*})$ and $(q_{1p}^{b*}, q_{2p}^{b*})$ are pessimistic Cournot equilibriums based on possibility distributions π_A and π_B. If $\pi_A \geq \pi_B$, then $[q_{1o}^{a*}, q_{2o}^{a*}] \geq [q_{1o}^{b*}, q_{2o}^{b*}]$ and $[q_{1p}^{a*}, q_{2p}^{a*}] \leq [q_{1p}^{b*}, q_{2p}^{b*}]$ holds.

The procedure for solving the equation (10)

Step 0. Arbitrarily choose a $a \in [a_c, a_r]$.

Step 1. Take $\hat{a}_{io}$ as a and solve Equation (10) to obtain (q_{1o}^*, q_{2o}^*).

Step 2. Calculate $e(a) = \pi_A(a) - u(w_i(a, q_{1o}^*, q_{2o}^*))$ If $|e(a)| \leq \varepsilon$ then $\hat{a}_{io} = a$ and stop; If $e(a) > \varepsilon$, then go to step 3; if $e(a) < -\varepsilon$, then go to step 4.

Step 3. Take a as $a = a + \Delta a (\Delta a > 0)$ and go back to step 1.

Step 4. Take a as $a = a - \Delta a (\Delta a > 0)$ and go back to step 1.

The procedure for solving the equation (11)

Step 0. Arbitrarily choose a $a \in [a_l, a_c]$.

Step 1. Take a as $\hat{a}_{ip}$ and solve Equation (11) to obtain (q_{1p}^*, q_{2p}^*).

Step 2. Calculate $e(a) = 1 - \pi_A(a) - u(w_i(a, q_{1p}^*, q_{2p}^*))$ If $|e(a)| \leq \varepsilon$ then $\hat{a}_{ip} = a$ and stop; If $e(a) > \varepsilon$, then go to step 3; if $e(a) < -\varepsilon$, then go to step 4.

Step 3. Take a as $a = a + \Delta a (\Delta a > 0)$ and go back to step 1.

Step 4. Take a as $a = a - \Delta a (\Delta a > 0)$ and go back to step 1.

II. COURNOT EQUILIBRIUM WITH ASYMMETRICAL POSSIBILISTIC INFORMATION

Let us consider asymmetrical information case. That is, some new information s on a can be used by firm 2 and not be used by firm 1. In this case, the output level of firm 2 is the function of s, that is, $q_2 = q_2(s)$. The profit of firms 1 and 2 are as follows:

$$w_1(a, q_1, q_2) = aq_1 - bq_1^2 - bq_1q_2(s) \tag{12}$$

$$w_2(a, q_1, q_2) = aq_2(s) - bq_2^2(s) - bq_1q_2(s) \tag{13}$$

For firm 1, the certain value of s is unknown but the possibility distribution of s is known. Based on extension principle, the possibility distribution of the output level of firm 2, denoted as $\pi_Q(q_2)$, can be calculated.

Definition 7. Conjecturing output level of firm 2 being q_2 and considering the possibility degree of q_2, the optimistic value of output level q_1 of firm 1 with information s, denoted as $Z_{1o}(q_1, q_2)$ is defined as follows:

$$Z_{1o}(q_1,q_2)=\max_a(\min(\pi_A(a),\pi_Q(q_2)\cdot u(w_1(q_1,q_2,a)))) \quad (14)$$

It can be seen from (14) that a higher evaluation value of q_1 is given by optimistic criterion if q_1 can produce a higher utility with higher possibility of a and q_2, which is of risk-preference.

Definition 8. Conjecturing output level of firm 2 being q_2 and considering the possibility degree of q_2, the pessimistic value of output level q_1 of firm 1 with information s, denoted as $Z_{1p}(q_1,q_2)$ is defined as follows:

$$Z_{1p}(q_1,q_2)=\min_a(\max(1-\pi_A(a),(1-\pi_Q(q_2))\cdot u(w_1(q_1,q_2,a)))) \quad (15)$$

It can be seen from (15) that a lower evaluation value of q_1 is given by pessimistic criterion if q_1 can produce a lower utility with higher possibility of a and q_2, which is of risk-aversion.

It is reasonable that firm 1 chooses its output level according to the following criteria.

(I) Firm 1 chooses its output level q_1 to maximize $Z_{1o}(q_1,q_2)$. That is,

$$q_1=\arg\max_{q_1} Z_{1o}(q_1,q_2)=h_{10}(q_2) \quad (16)$$

, which is a reaction function of firm 1, called optimistic reaction function with information s.

(II) Firm 1 chooses its output level q_1 to maximize $Z_{1p}(q_1,q_2)$. That is,

$$q_1=\arg\max_{q_1} Z_{1p}(q_1,q_2)=h_{1p}(q_2) \quad (17)$$

, which is a reaction function of firm 1, called pessimistic reaction function with information s.

Firm 2 can refine $\pi_A(a)$ by using additional information s. The renewed information is a kind of conditional possibility distribution, denoted as $\pi_{A|s}(a)$, which means that if information is s, then the possibility distribution of a is $\pi_{A|s}(a)$. Based on the renewed information, the optimal output level is decided by firm 2. For simplicity, it is supposed that $\pi_{A|s}(a)$ is a continuous function such that $\pi_{A|s}(a):[a_l,a_r]\to[0,1]$, where $\pi_{A|s}(a_c^{'})=1$, $\pi_{A|s}(a_l^{'})=0$ and $\pi_{A|s}(a_r^{'})=0$ （$a_l\le a_l^{'}<a_c^{'}<a_r^{'}\le a_r$）. $\pi_{A|s}(a)$ is an increasing function within $[a_l,a_c^{'}]$ and decreasing function within $[a_c^{'},a_r]$.

Definition 9. Conjecturing output level of firm 1 being q_1, based on conditional possibility distribution $\pi_{A|s}(a)$, the optimistic value of output level of firm 2, denoted as $Z_{2o}(q_1,q_2)$, is defined as follows:

$$Z_{2o}(q_1,q_2)=\max_a(\min(\pi_{A|s}(a),u(w_2(q_1,q_2,a)))) \quad (18)$$

Definition 10. Conjecturing output level of firm 1 being q_1, based on conditional possibility distribution $\pi_{A|s}(a)$, the pessimistic value of output level of firm 2, denoted as $Z_{2p}(q_1,q_2)$, is defined as follows:

$$Z_{2p}(q_1,q_2)=\min_a(\max(1-\pi_{A|s}(a),u(w_2(q_1,q_2,a)))) \quad (19)$$

It is reasonable that firm 2 chooses its output level according to the following criteria.

(I) Firm 2 chooses its output level q_2 to maximize $Z_{2o}(q_1,q_2)$ .Thai is,

$$q_2=\arg\max_{q_2} Z_{2o}(q_1,q_2)=k_{2o}(q_1) \quad (20)$$

, which is a reaction function of Firm 2, called optimistic reaction function with conditional possibility distribution.

(II) Firm 2 chooses its output level q_2 to maximize $Z_{2p}(q_1,q_2)$. That is,

$$q_2=\arg\max_{q_2} Z_{2p}(q_1,q_2)=k_{2p}(q_1) \quad (21)$$

, which is a reaction function of firm2, called pessimistic reaction function with conditional possibility distribution.

Definition 11. The solutions of the following equations (22) and (23), denoted as (q_{1o}^*,q_{2o}^*) and (q_{1p}^*,q_{2p}^*), are called as optimistic and pessimistic Cournot equilibriums with information s, respectively.

$$\begin{cases} q_1=h_{1o}(q_2) \\ q_2=k_{2o}(q_1) \end{cases} \quad (22)$$

$$\begin{cases} q_1=h_{1p}(q_2) \\ q_2=k_{2p}(q_1) \end{cases} \quad (23)$$

Proposition 4. The optimistic Cournot equilibrium with information s, (q_{1o}^*,q_{2o}^*), is the solution of the following equations.

$$\begin{cases} q_1=\dfrac{\hat{a}_{1o}(q_2)-bq_2}{2b} \\ q_2=\dfrac{\hat{a}_{2o}(q_1)-bq_1}{2b} \end{cases}, \quad (24)$$

where $\hat{a}_{1o}(q_2)$ is the horizontal coordinate of right intersection of $\pi_A(a)$ and $\pi_Q(q_2)\cdot u(w_1(a,q_1^*,q_2))$, which is called as optimistic focus point of firm 1 to represent the most considerable value by the optimistic viewpoint of firm 1. $\hat{a}_{2o}(q_1)$ is the horizontal coordinate of right intersection of

$\pi_{A|s}(a)$ and $u(w_2(a, q_1^*, q_2))$, which is called as optimistic focus point of firm 2 to represent the most considerable value by the optimistic viewpoint of firm 2.

Proposition 5. The pessimistic Cournot equilibrium (q_{1p}^*, q_{2p}^*) is the solution of the following equations.

$$\begin{cases} q_1 = \dfrac{\hat{a}_{1p}(q_2) - bq_2}{2b} \\ q_2 = \dfrac{\hat{a}_{2p}(q_1) - bq_1}{2b} \end{cases}, \tag{25}$$

where $\hat{a}_{1p}(q_2)$ is the horizontal coordinate of left intersection of $1-\pi_A(a)$ and $(1-\pi_Q(q_2)) \cdot u(w_1(a, q_1^*, q_2))$, which is called as pessimistic focus point of firm 1 to represent the most considerable value by the pessimistic viewpoint of firm 1. $\hat{a}_{2p}(q_1)$ is the horizontal coordinate of left intersection of $1-\pi_{A|s}(a)$ and $u(w_2(a, q_1, q_2^*))$, which is called as pessimistic focus point of firm 2 to represent the most considerable value by the pessimistic viewpoint of firm 2.

Theorem 4. There is one and only one optimistic Cournot equilibrium (q_{1o}^*, q_{2o}^*) in (24) and pessimistic Cournot equilibrium (q_{1p}^*, q_{2p}^*) in (25).

Proposition 6. The possibility distribution of profit of firm i is as follows:

$$\pi_W^i(w) = \pi_A\left(\frac{w + bq_i^{*2} + bq_i^* q_j^*}{q_i^*}\right),$$

(26)

where (q_1^*, q_2^*) is equilibrium outcomes of firms 1 and 2.

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