Robust Control Analysis for Uncertain Fuzzy Systems with Time-Delay

Jun Yoneyama
Department of Electronics and Electrical Engineering,
Aoyama Gakuin University,
6-16-1 Chitosedai, Setagaya, Tokyo 157-8572 Japan
E-mail: yoneyama@ee.aoyama.ac.jp,
Phone and Fax: +81-3-5384-6418

Abstract

This paper is concerned with quadratic stability and stability with disturbance attenuation of a class of uncertain fuzzy systems with time-delay. The class of systems under consideration is fuzzy time-delay systems with norm-bounded parameter uncertainties. Fuzzy systems often approximate nonlinear systems and thus uncertain fuzzy system description is useful to treat a wide class of systems. Moreover, uncertain systems with time delays generalize a wider class of nonlinear systems. Our main results are the relationship between robust H_∞ control via output feedback and a scaled H_∞ control problem and the one between quadratic stabilization and a standard H_∞ control problem for fuzzy systems with time-delay. These imply that quadratic stabilizing controllers for uncertain fuzzy systems with time-delay can be designed by solving H_∞ control problems for fuzzy systems with time-delay.

Keywords: Robust Control, Takagi-Sugeno Fuzzy Systems, Uncertain Systems, H_∞ Control, Quadratic Stability

1 Introduction

Control and modeling of nonlinear dynamical systems is one of the most challenging areas in systems and control theory. Nonlinear control has been active for many years, and many results concerning with nonlinear control have been obtained (for example, [4], [5]). However, it is still difficult to implement nonlinear controllers for practical systems. On the other hand, much effort has also been devoted to modeling nonlinear systems ([1]). System modeling and parameter identification of nonlinear dynamical systems are also important and difficult problems, and hence so is practical controller implementation for nonlinear systems. In order to overcome these difficulties, we pay much attention to robust control and its controller design. In the past years, there has been considerable attention to the problems of robust stabilization and robust performance of uncertain systems. One of the most representative problems in robust control theory is the so-called H_∞ control problem ([3], [13], [14], [15]). Recently, robust stabilization and robust H_∞ control problems for systems with uncertain parameters have been studied ([2], [6], [8], [9], [10]). Uncertainties in the systems come from the approximation of nonlinear dynamical systems by linear dynamical systems and identification error between original nonlinear systems and mathematical systems. Obviously, linear systems with uncertainties treat a wider class of systems and hence robust stabilization and robust H_∞ control problem for those uncertain systems are more practical problems. The solution to the robust H_∞ control problem was given using Riccati equation approach and LMI approach ([2], [6], [8], [9], [10]).

Time-delay systems often appear in industrial systems and information networks. Thus, it is important to analyze time-delay systems and design controllers for them. However, time-delay systems are, in general, infinite dimensional systems, which make the analysis and synthesis complicates. Recently, memoryless controllers have been successively employed to stabilize time-delay systems. Such controllers have been applied to fuzzy time-delay systems([11], [12]). Moreover, robust control for time-delay systems have been obtained in [7].

In this paper we consider quadratic stability and stability with disturbance attenuation for fuzzy time-delay systems with norm-bounded uncertain parameters. The uncertain parameters in the systems under consideration are additive structured uncertainties whose values are unknown but known to be bounded. We show relationships between quadratic stability with disturbance attenuation for uncertain fuzzy time-delay systems and a scaled H_∞ control for the same class of systems without uncertainties, and between quadratic stability and a standard H_∞ control. The relationships are shown through linear matrix inequalities. The paper is organized as follows; we introduce a class of systems with norm-bounded uncertain parameters and make some important definitions in Section 2. The robust H_∞ analysis and synthesis for uncertain time-delay systems are given in Section 3. Finally, concluding remarks are given in Section 4.

2 The Uncertain Systems

In this section, we introduce a class of fuzzy time-delay systems with norm-bounded parameter uncertainties and make some important definitions for the systems.

First, consider the Takagi-Sugeno fuzzy time-delay model described by the following IF-THEN rules:

$$\begin{array}{ll} \textbf{IF} & \xi_1 \text{ is } M_{i1} \text{ and } \cdots \text{ and } \xi_p \text{ is } M_{ip}, \\ \textbf{THEN} & \dot{x}(t) = A_i x(t) + A_{di} x(t-h) + B_i w(t), \\ & z(t) = C_i x(t) \quad i = 1, \cdots, r \end{array}$$

where h is an unknown time-delay, $x(t) \in \Re^n$ is the state, $w(t) \in \Re^{m_1}$ is the disturbance, $z(t) \in \Re^{q_1}$ is the controlled output, the matrices A_i, A_{di}, B_i and C_i are of appropriate dimensions, r is the number of IF-THEN rules, M_{ij} are fuzzy sets and $\xi_1, \cdots, \xi_p$ are premise variables. We set $\xi = \begin{bmatrix} \xi_1 & \cdots & \xi_p \end{bmatrix}^T$. Here we assume that the premise variables are given and do not depend on u. The state equation is defined as follows:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \lambda_i(\xi(t)) \{ A_i x(t) + A_{di} x(t-h) + B_i w(t) \}, \\ z(t) &= \sum_{i=1}^{r} \lambda_i(\xi(t)) C_i x(t) \end{aligned} \tag{1}$$

where

$$\lambda_i(\xi) = \frac{\beta_i(\xi)}{\sum_{i=1}^{r} \beta_i(\xi)}, \quad \beta_i(\xi) = \prod_{j=1}^{q} M_{ij}(\xi_j)$$

and $M_{ij}(\cdot)$ is the grade of the membership function of M_{ij}. We assume $\beta_i(\xi) \geq 0$, $i = 1, \cdots, r$, and $\sum_{i=1}^{r} \beta_i(\xi) > 0$, for any t. Hence $\lambda_i(\xi)$ satisfy $\lambda_i(\xi) \geq 0$, $i = 1, \cdots, r$ and $\sum_{i=1}^{r} \lambda_i(\xi) = 1$ for any t.

Definition 2.1 *The system (1) is said to be input-output stable if $z(t) \in L^2(0, \infty; \Re^{q_1})$ for any $w(t) \in L^2(0, \infty; \Re^{m_1})$ where $L^2(0, \infty; \cdot)$ is the space of square integrable functions.*

Definition 2.2 *Given a scalar $\gamma > 0$, the system (1) is said to be stable with disturbance attenuation γ if it is exponentially stable and input-output stable with*

$$\| z \|_{L_2}^2 \leq d^2 \| w \|_{L_2}^2 \tag{2}$$

for some $0 < d < \gamma$.

For the standard linear systems we have the following lemma.

Lemma 2.1 *Let $\gamma > 0$ be given, and consider the following system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-h) + Bw(t), \\ z(t) &= Cx(t). \end{aligned} \tag{3}$$

Then it is stable with disturbance attenuation γ if there exist matrices $X > 0$ and $Y > 0$ which satisfy

$$A^T X + XA + XWX + C^T C + Y = -P < 0 \tag{4}$$

where

$$W = \gamma^{-2} BB^T + A_d Y^{-1} A_d.$$

Proof: Take a Lyapunov function

$$V(x,t) = x^T(t) X x(t) + \int_{-h}^{0} x^T(t+\beta) Y x(t+\beta) d\beta \tag{5}$$

where X and Y are positive definite matrices satisfying (4). Note that since both X and Y are positive definite $V(x,t)$ is positive for all $x \neq 0$. We differentiate $V(x,t)$ with respect to t along the solution to (3);

$$\begin{aligned} \frac{dV}{dt} &= \dot{x}^T(t) X x(t) + x^T(t) X \dot{x}(t) + x^T(t) Y x(t) \\ &\quad - x^T(t-h) Y x(t-h) \\ &= x^T(t) [A^T X + XA + Y + C^T C \\ &\quad + X(\frac{1}{\gamma^2} BB^T + A_d Y^{-1} A_d^T) X] x(t) \\ &\quad - \gamma^2 |w(t) - \frac{1}{\gamma^2} B^T X x(t)|^2 \\ &\quad - |Y^{1/2}(x(t-h) - Y^{-1} A_d^T X x(t))|^2 \\ &\quad - (|z(t)|^2 - \gamma^2 |w(t)|^2) \\ &= -x^T(t) P x(t) - \gamma^2 |w(t) - \frac{1}{\gamma^2} B^T X x(t)|^2 \\ &\quad - |Y^{1/2}(x(t-h) - Y^{-1} A_d^T X x(t))|^2 \\ &\quad - (|z(t)|^2 - \gamma^2 |w(t)|^2). \end{aligned}$$

Since $P > \alpha I$ for some $\alpha > 0$, we have

$$\begin{aligned} \dot{V}(t) &< -|z(t)|^2 + \gamma^2 |w(t)|^2 - \alpha |x(t)|^2 \\ &\quad - \gamma^2 |w(t) - \frac{1}{\gamma^2} B^T X x(t)|^2 \\ &\quad - |Y^{1/2}(x(t-h) - Y^{-1} A_d^T X x(t))|^2. \end{aligned}$$

Integrating both sides from 0 to T, we have

$$\begin{aligned} & V(T) + ||z||_{2T}^2 + \alpha ||x||_{2T}^2 + \gamma^2 ||w - \tfrac{1}{\gamma^2} B^T X x||_{2T}^2 \\ & \quad + ||Y^{1/2}(x(t-h) - Y^{-1} E^T X x(t))||_{2T}^2 \\ & \leq \gamma^2 ||w||_{2T}^2 + V(0) \end{aligned}$$

where $\| \cdot \|_{2T}^2$ is the norm in $L^2(0, T; \cdot)$. Hence (3) is both exponentially stable and input-output stable. Moreover

$$\begin{aligned} & ||z||_{L_2}^2 + \alpha ||x||_{L_2}^2 + ||Y^{1/2}(x(t-h) - Y^{-1} E^T X x(t))||_{L_2}^2 \\ & \quad < \gamma^2 ||w||_{L_2}^2 \end{aligned}$$

which yields (2).

Now consider the Takagi-Sugeno fuzzy model with norm-bounded parameter uncertainties, described by the following fuzzy IF-THEN rules;

$$\begin{array}{ll} \text{IF} & \xi_1 \text{ is } M_{1i} \text{ and } \cdots \text{ and } \xi_p \text{ is } M_{pi}, \\ \text{THEN} & \dot{x}(t) = (A_i + \triangle A_i) x(t) \\ & \qquad + (A_{di} + \triangle A_{di}) x(t) + B_i w(t), \\ & z(t) = C_i x(t), \; i = 1, \cdots, r \end{array}$$

where $\triangle A_i, \triangle A_{di}$ represent the time-varying uncertain matrices of appropriate dimensions, and can describe the identification errors between the original systems and the local linear representation of the nonlinear systems. Then the state equation and the controlled output are defined as follows;

$$\begin{aligned} & \dot{x}(t) = (A(\lambda) + \triangle A(\lambda)) x(t) \\ & \qquad + (A_d(\lambda) + \triangle A_d(\lambda)) x(t) + B(\lambda) w(t), \\ & z(t) = C(\lambda) x(t) \end{aligned} \tag{6}$$

where

$$A(\lambda) = \sum_{i=1}^{r} \lambda_i(\xi(t))A_i,\ \Delta A(\lambda) = \sum_{i=1}^{r} \lambda_i(\xi(t))\Delta A_i,$$

$$A_d(\lambda) = \sum_{i=1}^{r} \lambda_i(\xi(t))A_{di}, \Delta A_d(\lambda) = \sum_{i=1}^{r} \lambda_i(\xi(t))\Delta A_{di},$$

$$B(\lambda) = \sum_{i=1}^{r} \lambda_i(\xi(t))B_i,\ C(\lambda) = \sum_{i=1}^{r} \lambda_i(\xi(t))C,$$

and similar notation will be used hereafter.

Definition 2.3 *Consider the system (6). The unforced uncertain system (6) with* $w = 0$ *is said to be quadratically stable if there exist matrices* $X > 0,\ Y > 0$ *such that*

$$\begin{array}{l}(A(\lambda) + \triangle A(\lambda))^T X + X(A(\lambda) + \triangle A(\lambda)) \\ \quad +X(A_d(\lambda) + \triangle A_d(\lambda))Y^{-1}(A_d(\lambda)A_d + \triangle A_d(\lambda))^T X \\ \quad +Y < 0\end{array}$$

for all admissible uncertainties $\triangle A, \triangle A_d$*. Similarly, the uncertain system (6) is said to be quadratically stabilizable via feedback controller if there exists a feedback controller such that the resulting closed-loop system is quadratically stable.*

3 Robust H_∞ Analysis and Synthesis

In this section, we show the relationships between robust H_∞ control and a scaled H_∞ control, and between quadratic stabilization and a standard H_∞ control problem.

Now consider the fuzzy model of Takagi and Sugeno described by the following fuzzy IF-THEN rules;

$$\begin{array}{ll}\text{IF} & \xi_1 \text{ is } M_{1i} \text{ and } \cdots \text{ and } \xi_p \text{ is } M_{pi}, \\ \text{THEN} & \dot{x}(t) = (A_i + \triangle A_i)x(t) \\ & \qquad +(A_{di} + \triangle A_{di})x(t-h) + B_{1i}w(t) \\ & \qquad +(B_{2i} + \triangle B_{2i})u(t), \\ & z(t) = C_{1i}x(t) + D_{12}u(t),\ i = 1, \cdots, r, \\ & y(t) = (C_{2i} + \triangle C_{2i})x(t) + D_{21i}w(t) \\ & \qquad +(D_{22i} + \triangle D_{22i})u(t)\end{array} \tag{7}$$

where $u(t) \in \Re^{m_2}$ is the control input, and $y(t) \in \Re^{q_2}$ is the observation. We assume that the uncertainties are of the form

$$\begin{array}{rl}\left[\begin{array}{cc}\triangle A_i & \triangle B_{2i} \\ \triangle C_{2i} & \triangle D_{22i}\end{array}\right] & = \left[\begin{array}{c}H_{1i} \\ H_{2i}\end{array}\right] F_i(t) \left[\begin{array}{cc}E_{1i} & E_{2i}\end{array}\right], \\ \triangle A_{di} & = H_{di}F_i(t)E_{di},\ i = 1, \cdots, r\end{array}$$

where $F_i(t) \in \Re^{l \times s},\ i = 1, \cdots, r$ are matrices of uncertain parameters such that

$$F_i^T(t)F_i(t) \leq I,\ i = 1, \ldots, r,$$

and $E_{1i}, E_{2i}, E_{di}, H_{1i}, H_{2i}$ and H_{di} are known real matrices of appropriate dimensions that characterize the structures of uncertainties.

Then the state equation and the controlled output are defined as follows;

$$\begin{array}{rcl}\dot{x}(t) & = & \sum_{i=1}^{r} \lambda_i(\xi(t))\{(A_i + H_{1i}F_iE_{1i})x(t) \\ & & +(A_{di} + H_{di}F_iE_{di})x(t-h) + B_{1i}w(t)+ \\ & & (B_{2i} + H_{1i}F_iE_{2i})u(t)\}, \\ z(t) & = & \sum_{i=1}^{r} \lambda_i(\xi(t))\{C_{1i}x(t) + D_{12i}u(t)\}, \\ y(t) & = & \sum_{i=1}^{r} \lambda_i(\xi(t))\{(C_{2i} + H_{2i}F_iE_{1i})x(t) \\ & & +D_{21i}w(t) + (D_{22i} + H_{2i}F_iE_{2i})u(t)\}.\end{array} \tag{8}$$

Suppose that the following rules concerning the output feedback controller for each subsystem (7) are given.

$$\begin{array}{ll}\textbf{IF} & \xi_1 \text{ is } M_{i1} \text{ and } \cdots \text{ and } \xi_p \text{ is } M_{ip}, \\ \textbf{THEN} & \dot{\hat{x}}(t) = \hat{A}_i\hat{x}(t) + \hat{B}_iy(t), \\ & u(t) = \hat{C}_i\hat{x}(t),\ i = 1, \cdots, r\end{array}$$

where all the matrices are of appropriate dimensions. Based on these rules, we take the following controller;

$$\begin{array}{rcl}\dot{\hat{x}}(t) & = & \sum_{i=1}^{r} \lambda_i(\xi(t))\{\hat{A}_i\hat{x}(t) + \hat{B}_iy(t)\}, \\ u(t) & = & \sum_{i=1}^{r} \lambda_i(\xi(t))\hat{C}_i\hat{x}(t),\end{array} \tag{9}$$

We use the same weights $\lambda_i(\xi(t))$ as those for the rules (7) of the fuzzy system.

Now we cooperate the H_∞ disturbance attenuation performance with the quadratic stability.

Definition 3.1 *Given a scalar* $\gamma > 0$*, the unforced uncertain system (8) with* $u(t) = 0$ *is said to be quadratically stable with disturbance attenuation* γ *if there exist matrices* $X > 0,\ Y > 0$ *such that for all admissible uncertainties* ΔA *and* ΔA_d*,*

$$\begin{array}{l}(A(\lambda) + \triangle A(\lambda))^T X + X(A(\lambda) + \triangle A(\lambda)) \\ \qquad +C^T(\lambda)C(\lambda) + XW(\lambda)X + Y < 0\end{array} \tag{10}$$

where

$$\begin{array}{l}W = \gamma^{-2}B_1(\lambda)B_1^T(\lambda) \\ \quad +(A_d(\lambda) + \triangle A_d(\lambda))Y^{-1}(A_d(\lambda) + \triangle A_d(\lambda))^T,\end{array}$$

and $(\cdot)(\lambda)$ *denotes* $\sum_{i=1}^{r} \lambda_i(\xi)(\cdot)$*. Similarly, the uncertain system (8) is said to be quadratically stabilizable with disturbance attenuation* γ *via output feedback controller if there exists an output feedback controller of the form (9) such that the resulting closed-loop system with (9) is quadratically stable with disturbance attenuation* γ*.*

Remark 3.1 *The notion of quadratic stability with disturbance attenuation is a natural extension of quadratic stability to incorporate* H_∞ *performance and its conservativeness lies in the requirement of fixed matrices* X *and* Y *in (10) for all admissible parameter uncertainties as in the quadratic stability. Despite its conservativeness, this notion naturally combines both quadratic stability and disturbance attenuation.*

Our main results will clarify the relationships between both robust H_∞ control via output feedback and a scaled H_∞ control problem and a standard H_∞ control problem for time-delay systems. In connection with the system (8) we now introduce a system below that will allow us to establish the relationship between robust H_∞ and a scaled H_∞ control problem.

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r}\lambda_i(\xi(t))\{A_i x(t) + A_{di}x(t-h) \\ &\quad + [\ \sqrt{\varepsilon}H_{1i}\quad \sqrt{\varepsilon}_d H_{di}\quad \gamma^{-1}B_{1i}\ ]\,\tilde{w}(t) \\ &\quad + B_{2i}u(t)\}, \\ \tilde{z}(t) &= \sum_{i=1}^{r}\lambda_i(\xi(t))\left\{\begin{bmatrix}\frac{1}{\sqrt{\varepsilon}}E_{1i}\\ \frac{1}{\sqrt{\varepsilon_d}}E_{di}\\ C_{1i}\end{bmatrix}x(t) + \begin{bmatrix}\frac{1}{\sqrt{\varepsilon}}E_{2i}\\ 0\\ D_{12i}\end{bmatrix}u(t)\right\}, \\ y(t) &= \sum_{i=1}^{r}\lambda_i(\xi(t))\{C_{2i}x(t) \\ &\quad + [\ \sqrt{\varepsilon}H_{2i}\quad 0\quad \gamma^{-1}D_{21i}\ ]\,\tilde{w}(t) + D_{22i}u(t)\}\end{aligned} \tag{11}$$

where $\tilde{w}(t) \in \Re^{2l+m_2}$ is the disturbance, $\tilde{z}(t) \in \Re^{2s+q_1}$ is the controlled outputs. The matrices $A_i, A_{di}, B_{1i}, B_{2i}, C_{1i}, C_{2i}, D_{12i}, D_{21i}, D_{22i}, E_{1i}, E_{di}, H_{1i}$ and H_{di} are the same as in the system (8), $\varepsilon > 0, \varepsilon_d > 0$ are parameters to be chosen and $\gamma > 0$ is the disturbance attenuation performance we wish to achieve for the system (8).

Lemma 3.1 *([2]) Suppose there exist $\varepsilon > 0$ and matrix $X > 0$ such that the following hold:*
a) $\varepsilon H^T X H < I$, and
b1) $A^T X A - X + \varepsilon A^T X H(I - \varepsilon H^T X H)^{-1}H^T X A + \frac{1}{\varepsilon}E^T E + C^T C < 0$, or
b2) $A^T(X^{-1} - \varepsilon H H^T)^{-1}A + \frac{1}{\varepsilon}E^T E + C^T C < 0$.
Then, we have

$$(A + HFE)^T X(A + HFE) - X + C^T C < 0 \tag{12}$$

for all F satisfying $F^T F \le I$.

Proof: Introducing

$$\begin{aligned}W_k = \varepsilon_d^{1/2}(I - \varepsilon_d H^T X H)^{-1/2}H^T X A \\ -\varepsilon_d^{-1/2}(I - \varepsilon_d H^T X H)^{1/2}F_d E\end{aligned}$$

we have

$$\begin{aligned}W_k^T W_k &= \varepsilon_d A^T X H(I - \varepsilon_d H^T X H)^{-1}H^T X A \\ &\quad - E^T F_{dk}^T H^T X A - A^T X H F_{dk} E \\ &\quad + \varepsilon_d^{-1} E^T F_{dk}^T (I - \varepsilon_d H^T X H) F_{dk} E \\ &= \varepsilon_d A^T X H(I - \varepsilon_d H^T X H)^{-1}H^T X A \\ &\quad - E^T F_{dk}^T H^T X A - A^T X H F_{dk} E \\ &\quad + \varepsilon_d^{-1} E^T F_{dk}^T F_{dk} E \\ &\quad - E^T F_{dk}^T H^T X H F_{dk} E \\ &\le \varepsilon_d A^T X H(I - \varepsilon_d H^T X H)^{-1}H^T X A \\ &\quad - E^T F_{dk}^T H^T X A - A^T X H F_{dk} E \\ &\quad + \frac{\rho_d^2}{\varepsilon_d}E^T E - E^T F_{dk}^T H^T X H F_{dk} E.\end{aligned}$$

Now considering a), we obtain

$$\begin{aligned}&\varepsilon_d A^T X H(I - \varepsilon_d H^T X H)^{-1}H^T X A + \frac{\rho_d^2}{\varepsilon_d}E^T E \\ &\ge E^T F_{dk}^T H^T X A + A^T X H F_{dk} E \\ &\quad + E^T F_{dk}^T H^T X H F_{dk} E\end{aligned}$$

Consequently, (12) follows from a) and b).

In view of Definition 3.1 and Lemma 3.1, Lemma 2.1 leads to the following theorem.

Theorem 3.1 *Given a constant $\gamma > 0$, the unforced system (11) with $u(t) = 0$ is quadratically stable with unitary disturbance attenuation if there exist common matrices $X > 0,\ Y > 0$ such that for some $\varepsilon, \varepsilon_d > 0$*

$$\begin{aligned}&Y - \frac{1}{\varepsilon_d}E_{di}^T E_{di} > 0, \\ &A_i^T X + X A_i + X W_i X + C_{1i}^T C_{1i} \\ &\qquad + \frac{1}{\varepsilon}E_{1i}^T E_{1i} + Y < 0,\ i = 1, \cdots, r\end{aligned} \tag{13}$$

where

$$\begin{aligned}W_i &= \gamma^{-2}B_{1i}B_{1i}^T + \varepsilon H_{1i}H_{1i}^T + \varepsilon_d H_{di}H_{di}^T \\ &\quad + A_{di}(Y - \frac{1}{\varepsilon_d}E_{di}^T E_{di})^{-1}A_{di}^T.\end{aligned}$$

In this case, the unforced system (11) with $u(t) = 0$ is quadratically stable with unitary disturbance attenuation. In this case, the unforced system (8) with $u(t) = 0$ is quadratically stable with disturbance attenuation γ

Proof: The conditions (13) turn out to be that there exist matrices $X > 0,\ \bar{Y} > 0$ satisfying

$$A_i^T X + X A_i + X W_i X + \bar{C}_i^T \bar{C}_i + Y < 0 \tag{14}$$

where

$$W_i = \bar{B}_i \bar{B}_i^T + A_{di} Y^{-1} A_{di}^T,$$

$$\bar{B}_i = [\ \sqrt{\varepsilon}H_{1i}\quad \sqrt{\varepsilon}_d H_{di}\quad \gamma^{-1}B_{di}\ ],$$

$$\bar{C}_i = \begin{bmatrix}\frac{1}{\sqrt{\varepsilon}}E_{1i}\\ \frac{1}{\sqrt{\varepsilon_d}}E_{di}\\ C_{1i}\end{bmatrix}.$$

It can be proved by Lemma 2.1 that the unforced system of (11) with $u(t) = 0$ is stable with unitary H_∞ disturbance attenuation if there exist matrices $X > 0,\ \bar{Y} > 0$ satisfying (14). Thus we show the first part of the theorem.

Using Lemma 3.1, we calculate the left-hand side of (4) for the system (8);

$$
\begin{aligned}
&(A(\lambda)+\triangle A(\lambda))^T X + X(A(\lambda)+\triangle A(\lambda))^T \\
&\quad + XW(\lambda)X + C^T(\lambda)C(\lambda) + Y \\
&\le \sum_{i=1}^{r} \lambda_i^2(\xi(t))[A_i^T X + XA_i + \frac{1}{\varepsilon}E_{1i}^T E_{1i} \\
&\qquad + \varepsilon X H_{1i}H_{1i}^T X + Y + C_{1i}^T C_{1i} + X\{\gamma^{-2}B_{1i}B_{1i}^T \\
&\qquad + \varepsilon_d H_{di}H_{di}^T + A_{di}(Y - \frac{1}{\varepsilon_d}E_{di}^T E_{di})^{-1}A_{di}^T\}X] \\
&+ \sum_{i<j}^{r} \lambda_i(\xi(t))\lambda_j(\xi(t))[A_i^T X + XA_i + \frac{1}{\varepsilon}E_{1i}^T E_{1i} \\
&\qquad + \varepsilon X H_{1i}H_{1i}^T X + Y + C_{1i}^T C_{1i} \\
&\qquad + X\{\gamma^{-2}B_{1i}B_{1i}^T + \varepsilon_d H_{di}H_{di}^T \\
&\qquad + A_{di}(Y - \frac{1}{\varepsilon_d}E_{di}^T E_{di})^{-1}A_{di}^T\}X \\
&\qquad + A_j^T X + XA_j + \frac{1}{\varepsilon}E_{1j}^T E_{1j} + \varepsilon X H_{1j}H_{1j}^T X \\
&\qquad + Y + C_{1j}^T C_{1j} + X\{\gamma^{-2}B_{1j}B_{1j}^T + \varepsilon_d H_{dj}H_{dj}^T \\
&\qquad + A_{dj}(Y - \frac{1}{\varepsilon_d}E_{dj}^T E_{dj})^{-1}A_{dj}^T\}X \\
&\qquad - (C_{1i}-C_{1j})^T(C_{1i}-C_{1j}) \\
&\qquad - X\{\gamma^{-2}(B_{1i}-B_{1j})(B_{1i}-B_{1j})^T \\
&\qquad - (A_{di}+H_{di}F_iE_{di}-A_{dj}-H_{dj}F_jE_{dj})Y^{-1} \\
&\qquad \times(A_{di}+H_{di}F_iE_{di}-A_{dj}-H_{dj}F_jE_{dj})^T\}X].
\end{aligned} \tag{15}
$$

The conditions (13) suffices to show (15) is negative definite, and thus it follows from Lemma 2.1 that we have the desired result.

The following theorem shows the relationship via output feedback control.

Theorem 3.2 *Let $\gamma > 0$ be a prescribed level of disturbance attenuation and the output feedback controller be of the form (9). Then the closed-loop system corresponding to (11) and (9) is stable with unitary disturbance attenuation if there exist matrices $\bar{X} > 0$, $\bar{Y} > 0$ such that*

$$
\begin{aligned}
&\bar{A}_{ijk}^T \bar{X} + \bar{X}\bar{A}_{ijk} + \bar{X}\bar{W}_{ij}\bar{X} + \bar{C}_{1ik}^T \bar{C}_{1ik} \\
&\qquad + \frac{1}{\varepsilon}\bar{E}_{1ik}^T \bar{E}_{1ik} + \bar{Y} < 0, \ i,j,k = 1,\cdots,r
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{W}_{ij} &= \gamma^{-2}\bar{B}_{1ij}\bar{B}_{1ij}^T + \varepsilon\bar{H}_{1ij}\bar{H}_{1ij}^T + \varepsilon_d\bar{H}_{di}\bar{H}_{di}^T \\
&\quad + \bar{A}_{di}(Y - \frac{1}{\varepsilon_d}\bar{E}_{di}^T E_{di})^{-1}\bar{A}_{di}^T,
\end{aligned}
$$

$$
\bar{A}_{ijk} = \begin{bmatrix} A_i & B_{2i}\hat{C}_k \\ \hat{B}_j C_{2i} & \hat{A}_j + \hat{B}_j D_{22i}\hat{C}_k \end{bmatrix},
$$

$$
\bar{A}_{di} = \begin{bmatrix} A_{di} & 0 \\ 0 & 0 \end{bmatrix},
$$

$$
\bar{H}_{1ij} = \begin{bmatrix} H_{1i} \\ \hat{B}_j H_{2i} \end{bmatrix}, \ \bar{H}_{di} = \begin{bmatrix} H_{di} \\ 0 \end{bmatrix},
$$

$$
\bar{E}_{1ik} = \begin{bmatrix} E_{1i} & E_{2i}\hat{C}_k \end{bmatrix}, \ \bar{E}_{di} = \begin{bmatrix} E_{di} & 0 \end{bmatrix},
$$

$$
\bar{B}_{1ij} = \begin{bmatrix} B_{1i} \\ \hat{B}_j D_{21i} \end{bmatrix}, \ \bar{C}_{1ik} = \begin{bmatrix} C_{1i} & D_{12i}\hat{C}_k \end{bmatrix}.
$$

In this case, the system (8) is quadratically stabilizable with disturbance attenuation γ via the output feedback controller (9).

Proof: We take the controller (9). the closed-loop system (11) with (9) becomes

$$
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\lambda_i(\xi(t))\lambda_j(\xi(t))\lambda_k(\xi(t)) \\
&\quad \times\{\bar{A}_{ijk}\bar{x}(t) + \bar{A}_{di}\bar{x}(t-h) \\
&\quad + \begin{bmatrix} \sqrt{\varepsilon}\bar{H}_{1ij} & \sqrt{\varepsilon}_d\bar{H}_{di} & \gamma^{-1}\bar{B}_{1ij} \end{bmatrix}\tilde{w}(t)\}, \\
z(t) &= \sum_{i=1}^{r}\sum_{k=1}^{r}\lambda_i(\xi(t))\lambda_k(\xi(t)) \begin{bmatrix} \frac{1}{\sqrt{\varepsilon}}\bar{E}_{1ik} \\ \frac{1}{\sqrt{\varepsilon}_d}\bar{E}_{di} \\ \bar{C}_{1ik} \end{bmatrix}\bar{x}(t)
\end{aligned}
$$

where $\bar{A}_{ijk}, \bar{A}_{di}, \bar{B}_{1ij}, \bar{H}_{1ij}, \bar{H}_{di}, \bar{E}_{1ik}, \bar{E}_{di}$ and $\bar{C}_{1ik}$ are the same as above. Also, it follows that the closed-loop system (8) with (9) is given by the form

$$
\begin{aligned}
\dot{\bar{x}}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\lambda_i(\xi(t))\lambda_j(\xi(t))\lambda_k(\xi(t)) \\
&\quad \times\{(\bar{A}_{ijk} + \bar{H}_{1ij}F_i\bar{E}_{1ik})\bar{x}(t) \\
&\quad + (\bar{A}_{di} + \bar{H}_{di}F_i\bar{E}_{di})\bar{x}(t-h) + \bar{B}_{1ij}w(t)\}, \\
z(t) &= \sum_{i=1}^{r}\sum_{k=1}^{r}\lambda_i(\xi(t))\lambda_k(\xi(t))\bar{C}_{1ik}\bar{x}(t)
\end{aligned}
$$

where $\bar{x}^T = \begin{bmatrix} x^T & \hat{x}^T \end{bmatrix}$. The desired result follows immediately from Theorem 3.1.

Remark 3.2 *Theorem 3.2 established the relationship between the robust H_∞ control problem for the fuzzy time-delay system (8) and the scaled H_∞ control problem for the system (11). Therefore, a solution to the robust H_∞ control problem for fuzzy time-delay systems can be obtained via existing H_∞ control techniques([13]) for the same class of systems. Moreover, Theorem 3.2 allows us to obtain output feedback controllers that solve the robust H_∞ control problem.*

Next, we discuss the quadratic stability and quadratic stabilization problems of fuzzy time-delay system (8). Similar to the robust H_∞ control, the quadratic stability and quadratic stabilization of the system (8) will be shown to be related to the H_∞ control of the following system:

$$
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r}\lambda_i(\xi(t))\{A_ix(t) + A_{di}x(t-h) \\
&\quad + \begin{bmatrix} \sqrt{\varepsilon}H_{1i} & \sqrt{\varepsilon}_d H_{di} \end{bmatrix}\tilde{w}(t) + B_{2i}u(t)\}, \\
\tilde{z}(t) &= \sum_{i=1}^{r}\lambda_i(\xi(t))\left\{\begin{bmatrix} \frac{1}{\sqrt{\varepsilon}}E_{1i} \\ \frac{1}{\sqrt{\varepsilon_d}}E_{di} \end{bmatrix}x(t)\right. \\
&\qquad \left. + \begin{bmatrix} \frac{1}{\sqrt{\varepsilon}}E_{2i} \\ 0 \end{bmatrix}u(t)\right\}, \\
y(t) &= \sum_{i=1}^{r}\lambda_i(\xi(t))\{C_{2i}x(t) \\
&\quad + \begin{bmatrix} \sqrt{\varepsilon}H_{2i} & 0 \end{bmatrix}\tilde{w}(t) + D_{22i}u(t)\}
\end{aligned} \tag{16}
$$

where $x(t) \in \Re^n$ is the state, $\tilde{w}(t) \in \Re^{2l}$ is the disturbance, $\tilde{z}(t) \in \Re^{2s}$ is the controlled output. The matrices

$A_i, A_{di}, B_{2i}, C_{2i}, D_{22i}, E_{1i}, E_{di}, H_{1i}$ and H_{di} are the same as in the system (8). The relationships between quadratic stability and H_∞ control, and between quadratic stabilization and H_∞ control are given in the following theorems.

Theorem 3.3 *The unforced system (16) with* $u(t) = 0$ *is stable with unitary disturbance attenuation if there exist common matrices* $X > 0,\ Y > 0$ *such that for some* $\varepsilon, \varepsilon_d > 0$

$$\begin{aligned} & Y - \frac{1}{\varepsilon_d} E_{di}^T E_{di} > 0, \\ & A_i^T X + X A_i + X W_i X + C_{1i}^T C_{1i} \\ & \qquad + \frac{1}{\varepsilon} E_{1i}^T E_{1i} + Y < 0,\ i = 1, \cdots, r \end{aligned} \tag{17}$$

where

$$\begin{aligned} W_i &= \varepsilon H_{1i} H_{1i}^T + \varepsilon_d H_{di} H_{di}^T \\ &\quad + A_{di}(Y - \frac{1}{\varepsilon_d} E_{di}^T E_{di})^{-1} A_{di}^T. \end{aligned}$$

In this case, the unforced system (8) with $u(t) = 0$ *is quadratically stable.*

Proof: Similar to proof of Theorem 3.1, it can be shown that (17) is a sufficient condition for stability with unitary disturbance attenuation for the system (16). Moreover, if there exists a common positive definite matrix X such that (17) is satisfied, then we have

$$(A(\lambda) + \triangle A(\lambda))^T X + X(A(\lambda) + \triangle A(\lambda)) < 0,$$

which implies that the unforced system (8) with $u(t) = 0$ is quadratically stable.

Theorem 3.4 *The closed-loop system corresponding to (16) and (9) is stable with unitary disturbance attenuation if there exist matrices* $\bar{X} > 0,\ \bar{Y} > 0$ *such that*

$$\begin{aligned} & \bar{A}_{ijk}^T \bar{X} + \bar{X} \bar{A}_{ijk} + \bar{X} \bar{W}_{ij} \bar{X} + \bar{C}_{1ik}^T \bar{C}_{1ik} \\ & \qquad + \frac{1}{\varepsilon} \bar{E}_{1ik}^T \bar{E}_{1ik} + \bar{Y} < 0,\ i, j, k = 1, \cdots, r \end{aligned} \tag{18}$$

where

$$\begin{aligned} \bar{W}_{ij} &= \varepsilon \bar{H}_{1ij} \bar{H}_{1ij}^T + \varepsilon_d \bar{H}_{di} \bar{H}_{di}^T \\ &\quad + \bar{A}_{di}(Y - \frac{1}{\varepsilon_d} \bar{E}_{di}^T E_{di})^{-1} \bar{A}_{di}^T, \end{aligned}$$

In this case, the system (8) is quadratically stabilizable via the output feedback controller (9).

Proof: Similar to proof of Theorem 3.2, it can be shown that the closed-loop system (16) with (9) is stable with unitary disturbance attenuation, and the closed-loop system (8) with (9) is quadratically stable if there exists a common positive definite matrix $\bar{X}$ such that (18) is satisfied.

Remark 3.3 *In view of Theorem 3.3, it results that the separation principle for* H_∞ *control also carries over to quadratic stabilization of (8) via output feedback. Moreover, Theorem 3.3 also allows us to solve output feedback controllers that quadratically stabilize (8).*

4 Conclusion

For fuzzy time-delay systems, we have shown the relationships between robust H_∞ control and a scaled H_∞ control, and between quadratic stabilization and a standard H_∞ control problem, which allows us to solve the robust controllers for uncertain fuzzy time-delay systems.

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